

On closed currents invariant by holomorphic foliations, I

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1 Introduction

There are several motivations to pay attention to singular holomorphic foliations defined on complex surfaces and admitting an invariant closed positive current. First the study of these foliations has a natural ergodic theoretic interest. Additional motivation comes from observing that this class of foliations captures and unifies special types of foliations such as those having a compact leaf, foliations having a Zariski-dense leaf isomorphic to a quotient of \mathbb{C} and Hilbert modular foliations. It follows that the understanding of these foliations has consequences in a variety of domains.

The purpose of this paper is to set up a dynamical method to study this type of foliations. The paper is essentially elementary in that only the basic properties of this method are considered in detail. Nonetheless, it will soon be apparent that the problem splits naturally into two cases corresponding to the two sources of known examples of foliations admitting invariant closed currents, namely foliations having a compact leaf and foliations carrying a transverse riemannian structure. Once this splitting will have been established in our context, we shall pursue the case leading to the existence of a compact leaf. The other case, for which more subtle dynamical arguments will be required, is going to be treated in the future.

Let us give a brief description of the point of view adopted here. The starting point is a construction appearing in [B-L-M] which is revisited in Section 2.3. Essentially the construction consists of observing that, inside the leaves of \mathcal{F} , there are real one-dimensional trajectories along to the which the holonomy of \mathcal{F} “tends to be contractive” so long they stay away from the singularities of \mathcal{F} . These trajectories can be viewed as defining a singular real one-dimensional foliation denoted by \mathcal{H} (cf. Section 2.3 for details). Similar ideas already appeared in the context of foliated characteristic classes, see [G] and its references. The natural idea is then to “follow” these trajectories and to understand how an invariant closed positive current can be compatible with the resulting contraction of the holonomy of \mathcal{F} . However, when this idea is further elaborated, the above mentioned dichotomy manifests itself in the existence or absence of trajectories of \mathcal{H} contained in the support of the current in question and possessing “infinite length” (cf. Section 5). If trajectories of infinite length do exist, then they must yield some “definite amount of contraction” in the holonomy pseudo-group of \mathcal{F} . This clearly poses a serious obstruction to the existence of the mentioned currents. The tension between the existence of the invariant current and the existence of “contractions” in the holonomy pseudogroup is likely to imply that the invariant current has a trivial nature: it is concentrated on a compact leaf of \mathcal{F} . On the other hand, it may well happen that all the trajectories of \mathcal{H} , or at least those contained in the support of the current, are of “finite length”. In vague terms, this means that every trajectory has “extremities” or “ends” so that they cannot be followed “for an arbitrarily large period of time”. This prevents us from ensuring the existence of “contractions” in the holonomy

pseudo-group of \mathcal{F} . Simple examples where this phenomenon is observed arise in the context of transversely riemannian foliations whose discussion will be postponed to a subsequent work.

The contents of this article is then somehow bipartite. It begins with the general definition of the foliation \mathcal{H} consisting of the mentioned real trajectories and it continues with the analysis of their behavior near the singularities of \mathcal{F} that may exist in the support of a (invariant closed positive) current. Then we turn to the global geometry of these trajectories. After some general reductions and definitions, we shall finally be confronted with the basic dichotomy mentioned above: either there are trajectories inside the support of the current having “infinite length” or all these trajectories have finite length.

The remainder of the paper will then be devoted to the investigation of the first possibility, i.e. of the case in which there are trajectories of infinite length contained in the support of our current. The corresponding results are ultimately summarized by Theorem 6.1 (see also Theorem A below). The statement of this theorem is as follows: unless all the \mathcal{H} -trajectories contained in the support of the current have a uniformly bounded length, then this support contains a compact leaf of \mathcal{F} . Although the terminology is at this point imprecise, we can made explicit the contents of this theorem as follows:

- at least as far as compact leaves are concerned, we only need to study those foliations for which there is a compact invariant set where all the \mathcal{H} -trajectories have finite length.
- if, by some reason, we can guarantee the existence of \mathcal{H} -trajectories of infinite length in the support of an invariant closed current T , then this support contains a compact leaf.

As to the second item, some non-trivial examples in which it is possible to prove *a priori* that no trajectory of \mathcal{H} on M is of finite length will be supplied at the very end of this article. These examples include some foliations on elliptic $K3$ surfaces having singularities that either are hyperbolic or belong to the domain of Siegel. It will be seen that for these examples, the cohomology class of an invariant closed current T giving no mass to individual leaves must have trivial self-intersection (and thus it has the same cohomology of an elliptic fiber). Nonetheless it is not *a priori* clear that the foliation in question needs to have any compact leaf at all. Another interesting class of examples includes foliations in the projective plane having singularities that may contribute non-trivially to the Lelong numbers of T , so as to allow the foliated current T to have strictly positive self-intersection. For example, consider a foliation \mathcal{F} on $\mathbb{CP}(2)$ having a radial singularity p and such that their remaining singularities belong to the Siegel domain or are hyperbolic. By a radial singularity, it is meant that \mathcal{F} is given in suitable coordinates about p by the 1-form $x dy - y dx$. If the degree of this foliation is at least 3, then we can arrange for the trajectories of \mathcal{H} to have infinite length. Similarly if the degree of the foliation is at least 4 then the same result applies to foliations having up to 2 radial singularities (or more generally two singularities whose eigenvalues are $1, \lambda$ with $\lambda \in \mathbb{R}_+^*$). Thus the results of this paper can be applied to these foliations to yield, in particular, the existence of algebraic curves invariant for them. More details on the construction of examples can be found at the end of Section 7.

The study of the dynamics of these “contractive trajectories” is likely to have interest in different problems about holomorphic foliations. For example, it may be useful to study the dynamics of “generic” foliations such as in [CDFG]. Similarly it should be mentioned that B. Deroin and V. Kleptsyn have employed the foliated Brownian motion to study the transverse dynamics of a conformal Riemann surface lamination, [D-K]. Roughly speaking they show the evolution of “most points” under the Brownian motion tends to give rise to a “contractive holonomy”. In

this sense their work seems to be related to ours, i.e. the Brownian motion evolution seems to be related to the foliation \mathcal{H} . It would be interesting to clarify possible relations between these approaches. Along these lines, the first sections of this paper may also serve as an introduction to this circle ideas. In addition, we have included a preliminary section providing details on some well-known facts that are usually not detailed in the literature. In particular the construction of [B-L-M] is reviewed. We also explain in detail the role played by the condition of having an ambient surface that is algebraic as well as the relation between Dirac masses for transversely invariant measures and compact leaves for singular foliations. Hopefully this discussion will be useful for readers that are not experts in foliation theory.

Let us now state

Theorem A (Main Theorem): *Let \mathcal{F} be a singular holomorphic foliation on an algebraic surface M . Suppose that \mathcal{F} carries an invariant closed positive current T and let \mathcal{K} denote a (singular) minimal set for \mathcal{F} contained in the support of T . Suppose also that in \mathcal{K} there is one trajectory of \mathcal{H} having infinite length. Then \mathcal{K} consists of an algebraic curve left invariant by \mathcal{F} .*

Let us point out that the above theorem *does not* state that the current T coincides with the integration current over the mentioned algebraic curve (at least on a neighborhood of \mathcal{K}). A partial answer to this question is however supplied by Theorem B below. The reader can check Section 7 for the definition of Liouvillean integrability.

Theorem B (Complement to Main Theorem): *With the notations of Theorem A, suppose that T is not locally given by integration over \mathcal{K} . Then \mathcal{F} admits a Liouvillean first integral on a neighborhood of \mathcal{K} .*

A by-product of the method developed here is the existence of a “contractive” element in its holonomy pseudogroup provided that \mathcal{H} has a trajectory of infinite length on M . In principle, this contractive element may be either a local hyperbolic diffeomorphism or a “ramified” super-attractive contraction (cf. Section 6 for further details). Also a type of “exponential super-contraction” may appear in connection with saddle-nodes. This is proven here under the additional assumption that the singularities of \mathcal{F} yield no saddle-node singularities under the reduction procedure of Seidenberg (cf. Section 2, singularities verifying this condition are sometimes called “generalized curves”). The presence of saddle-node in the picture would actually not affect neither our methods nor the validity of the conclusion. Nonetheless the elimination of this “superfluous” assumption would require us to discuss the behavior of the mentioned trajectories around a saddle-node and, in turn, this would lead us to a long detour in our way to the goals of the present paper. Yet, the study of these trajectories near saddle-node singularities is an essential part of the analysis involved in the continuation of this work, so it seems natural to let the general result about “existence of contractions” to be complemented there. We note however that, under the above assumption concerning saddle-nodes singularities, the methods used here ensure the existence of contraction given either by a local hyperbolic diffeomorphism or by a “ramified” super-attractive map, cf. Section 6.

To close this Introduction, let us briefly outline the contents of this article. Section 2 contains background material on the subject. In Section 2.1 we recall Seidenberg’s reduction procedure for the singularities of holomorphic foliations. Section 2.2 contains precise definitions and a few basic facts regarding closed currents invariant by a foliation including its relation with the notion of transverse invariant measure. Finally, in Section 2.3, the main ideas of [B-L-M] are presented. In particular the foliation \mathcal{H} (associated to a given holomorphic foliation \mathcal{F}) is defined.

Section 3 is devoted to a detailed study of the behavior of \mathcal{H} on a neighborhood of a singularity of \mathcal{F} belonging to the Siegel domain. This includes a discussion of the “Dulac transform” defined by means of \mathcal{H} . Building on the material presented in Section 3, we develop in Section 4 a detailed analysis of the singularities of \mathcal{F} lying in the support of T . The structure of these singularities will play an important role in the proof of Theorem 6.1 which is a slightly more general version of Theorem A.

Section 5 begins with a few global definitions, in particular the precise definition of “trajectory of \mathcal{H} ” and of its corresponding length. This section is technically very simple and its main result is summarized by Proposition 5.8. Finally in Sections 6 and 7, we shall use the preceding material to deduce the proofs of the above stated theorems.

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2 Preliminaries

2.1 Generalities about foliations on algebraic surfaces

Let M be a smooth compact complex surface. A *singular holomorphic foliation* \mathcal{F} on M consists of the following data:

1. An open covering $\{U_i\}$ of M .
2. A holomorphic vector field Y_i with isolated singularities defined on U_i for each i .
3. For each pair i, j such that $U_i \cap U_j \neq \emptyset$ a function $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ such that $Y_i = g_{ij}Y_j$.

In particular it follows that the singularities of \mathcal{F} correspond to those of the Y_i ’s and are therefore isolated. The foliation \mathcal{F} can alternatively be defined through holomorphic forms ω_i (rather than vector fields Y_i) subjected to the relations $\omega_i = f_{ij}\omega_j$ with $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$. The transition functions g_{ij} (resp. f_{ij}) satisfy the natural cocycle relations and hence give rise to a line bundle $T^*\mathcal{F}$ (resp. $N\mathcal{F}$) on M which is called the cotangent bundle of \mathcal{F} (resp. normal bundle of \mathcal{F}). Furthermore, by virtue of the relations $Y_i = g_{ij}Y_j$, the foliation \mathcal{F} can be interpreted as a global holomorphic section of $T^*\mathcal{F} \otimes TM$ with discrete zero set and modulo multiplication by non-vanishing holomorphic functions. Thus we obtain.

Lemma 2.1 *A singular holomorphic foliation on an algebraic surface M is always given by a globally defined meromorphic form ω (resp. meromorphic vector field Y). Besides we can suppose without loss of generality that the meromorphic form ω is not closed.*

Proof. The fact that M is algebraic guarantees that every line bundle over M admits non-trivial meromorphic sections. In turn, this shows that \mathcal{F} is generated by a global *meromorphic* vector field (or differential 1-form) in the obvious sense. Finally, if \mathcal{F} is given by a closed form ω , then we just need to replace ω by $f\omega$ for a generic meromorphic function f . In fact, we have

$$d(f\omega) = df \wedge \omega + f d\omega = df \wedge \omega \neq 0$$

for a “generic” f . This proves the lemma. □

Remark 2.2 General Setting: Throughout this work M is supposed to be a complex surface equipped with a holomorphic foliation \mathcal{F} which is given by a non-closed meromorphic form ω . The preceding lemma shows that this is always the case for M projective algebraic. In the latter case, it would also be possible to choose a 1-form ω satisfying further “generic conditions”. Suitable generic properties would simplify some parts of our discussion but we decided not to use this. The main reasons for our choice, besides having a slightly more general result, lies in the fact that some “generic properties of ω ” does not allow us, for example, to avoid a non-trivial intersection of $(\omega)_0$ and $(\omega)_\infty$ at a regular point of \mathcal{F} . To eliminate these intersection points a natural idea is to blow them up what, in turn, would bring us back to a situation where the corresponding transform of ω is no longer “generic”. Thus this transform would need to be replaced by a generic 1-form on the blown-up surface and the final construction would (even if successful) rely on constructions and arguments of algebraic geometry that appear to me as less elementary than the approach chosen here. Indeed I also think that the treatment of all the “degenerate situations” that may arise for an arbitrary ω ends up making the argument more “concrete”.

The fact that \mathcal{F} is generated by global meromorphic differential forms will be exploited in Paragraph 2.3. For the time being, we are going to focus on local aspects such as the structure of the singularities of \mathcal{F} . We can then suppose that \mathcal{F} is given on a neighborhood of $(0, 0) \in \mathbb{C}^2$ by a holomorphic 1-form $\eta = Pdy + Qdx$ having an isolated singularity at the origin. Sometimes it is also useful to think of \mathcal{F} as being given by the vector field $Y = P\partial/\partial x - Q\partial/\partial y$. The *order* of \mathcal{F} at $(0, 0) \in \mathbb{C}^2$ is by definition the order of the first non-zero jet of η at $(0, 0) \in \mathbb{C}^2$. This notion is well-defined since η (or Y) has isolated singularities. Similarly we define the eigenvalues of \mathcal{F} at $(0, 0)$ as the eigenvalues of the linear part of Y at $(0, 0)$. These eigenvalues are therefore defined up to a multiplicative constant so that only their quotient has an intrinsic meaning.

Let λ_1, λ_2 be the eigenvalues of \mathcal{F} at $(0, 0)$. We say that $(0, 0)$ is a *hyperbolic* singularity if $\lambda_1\lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$. If $\lambda_1\lambda_2 \neq 0$ but $\lambda_1/\lambda_2 \in \mathbb{R}_-$, then we say that $(0, 0)$ is in the *Siegel domain*. The singularity is said to be a *saddle-node* if $\lambda_1 \neq 0$ and $\lambda_2 = 0$.

Singularities whose eigenvalues satisfy $\lambda_1\lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \in \mathbb{R}_+$ need some specific attention. Let us begin by saying that a singularity (whose both eigenvalues are possibly zero) is *dicritical* if it admits infinitely many *separatrices* i.e. analytic curves passing through the singularity and invariant under the foliation. Now consider a foliation whose eigenvalues λ_1, λ_2 at the origin verify $\lambda_1\lambda_2 \neq 0$ and $\lambda_1 = n\lambda_2$ for $n \in \mathbb{N}$ (or $\lambda_2 = n\lambda_1$). This foliation is then conjugate to the foliation given either by the 1-form

$$nx dy - y dx$$

or by the 1-form

$$(nx + y^n) dy - y dx .$$

In the first case $(0, 0)$ is dicritical. In the second case it is said to be a Poincaré-Dulac singularity (cf. [A-Y]). Next, if $\lambda_1\lambda_2 \neq 0$, $\lambda_1/\lambda_2 \in \mathbb{R}_+$ but λ_1/λ_2 is not an integer nor the inverse of an integer, then Poincaré Theorem asserts that \mathcal{F} is linearizable (cf. [A-Y]). In other words, \mathcal{F} is conjugate to the foliation given by

$$\lambda_1 x dy - \lambda_2 y dx .$$

The preceding implies that a singularity with eigenvalues $\lambda_1\lambda_2 \neq 0$ is dicritical if and only if it is not a Poincaré-Dulac singularity and $\lambda_1/\lambda_2 \in \mathbb{Q}_+$ and $(0, 0)$. When $\lambda_1/\lambda_2 \in \mathbb{R}_+ \setminus \mathbb{Q}$ the resulting singularity is going to be called an *irrational focus*.

Next we need to recall Seidenberg's reduction of singularities theorem [Sei]. Let $\pi : \tilde{\mathbb{C}}^2 \rightarrow \mathbb{C}^2$ denote the blow-up of \mathbb{C}^2 at the origin. If \mathcal{F} is defined on a neighborhood U of $(0, 0) \in \mathbb{C}^2$, then $\pi^*\mathcal{F}$ naturally defines a holomorphic foliation on $\pi^{-1}(U)$. The foliation $\pi^*\mathcal{F} = \tilde{\mathcal{F}}_1$ is called the blow-up of \mathcal{F} . Clearly this construction can be iterated: if p is a singularity of $\tilde{\mathcal{F}}_1$, then $\tilde{\mathcal{F}}_1$ can be blown up at p to provide a new foliation defined on an appropriate open surface. Seidenberg's theorem [Sei] then claims the existence of a finite sequence of blow-ups

$$\mathcal{F} = \mathcal{F}_0 \xleftarrow{\pi_1} \tilde{\mathcal{F}}_1 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_n} \tilde{\mathcal{F}}_n$$

such that the following holds:

- The irreducible components of the (total) exceptional divisor $(\pi_1 \circ \dots \circ \pi_n)^{-1}(0)$ are smooth rational curves D_1, \dots, D_n of strictly negative self-intersection.
- The singularities of $\tilde{\mathcal{F}}_n$ are *reduced* i.e. they are of one of the following types: hyperbolic, in the Siegel domain, saddle-node or an irrational focus.

It should be noted that the exceptional divisor $(\pi_1 \circ \dots \circ \pi_n)^{-1}(0)$ need not be invariant by $\tilde{\mathcal{F}}_n$. In fact, it may contain irreducible components invariant under $\tilde{\mathcal{F}}_n$ along with irreducible components that are not invariant by $\tilde{\mathcal{F}}_n$. If D_i is an irreducible component that is not invariant by $\tilde{\mathcal{F}}_n$, then the projection of the regular leaves of $\tilde{\mathcal{F}}_n$ transverse to D_i produces infinitely many separatrices for the initial foliation $\mathcal{F} = \mathcal{F}_0$. In other words, \mathcal{F} is dicritical. Conversely if \mathcal{F} is dicritical then, in the above situation, there must exist at least one irreducible component of $(\pi_1 \circ \dots \circ \pi_n)^{-1}(0)$ which is not invariant by $\tilde{\mathcal{F}}_n$. The components of $(\pi_1 \circ \dots \circ \pi_n)^{-1}(0)$ that are not invariant by $\tilde{\mathcal{F}}_n$ are also said to be *dicritical*.

2.2 Closed currents and transverse invariant measures

Let us now recall some standard definitions and results concerning closed invariant currents and transverse invariant measures for singular foliations.

Given M as before, we denote by $D^p(M)$ the Fréchet space of C^∞ -differential forms on M of degree p . The space of currents of *dimension* p is, by definition, the topological dual $D'_p(M)$ of $D^p(M)$. The space of currents possesses a natural differential “ d ” (as well as operators $\partial, \bar{\partial}$) obtained by duality from the usual operators acting on differential forms. In particular, we can talk about closed/exact currents.

Suppose now that M is endowed with a (singular) foliation \mathcal{F} . Consider a current T of dimension 2 and denote by $\text{supp}(T) \subseteq M$ its support. The current T is said to be *invariant* by \mathcal{F} if $T(\beta) = 0$ for every 2-form β vanishing on \mathcal{F} . An invariant current is sometimes also called a *foliated current* or a *current directed by \mathcal{F}* . Because M is a complex surface and \mathcal{F} is a holomorphic foliation, every current T as above is of type $(1, 1)$. In fact, on a neighborhood of a regular point, we can choose coordinates (x, y) in which \mathcal{F} is given by $dx = 0$. Hence a 2-form β vanishing on \mathcal{F} must be given by $\alpha_x \wedge dx + \alpha_{\bar{x}} d\bar{x}$ where $\alpha_x, \alpha_{\bar{x}}$ are appropriate 1-forms. Now the invariance of T under \mathcal{F} becomes encoded in the normal form

$$T = T(x, y) dx \wedge d\bar{x}$$

where $T(x, y)$ is naturally identified with a distribution. This shows that T is a $(1, 1)$ -current as claimed. For these currents, the notion of being *positive* becomes especially transparent: T

is said to be positive if the local coefficient $T(x, y)$ is identified with a positive measure. An equivalent condition consists of saying that for every smooth $(1, 0)$ -form α the wedge product $T \wedge i\alpha \wedge \bar{\alpha}$ is a positive measure on M .

The most basic example of a foliation admitting a closed positive current is provided by a compact leaf of a foliation \mathcal{F} . More precisely, let C be a compact curve (smooth to simplify) which is invariant by \mathcal{F} . Consider then the current of integration over C , namely the current T given by

$$T(\beta) = \int_C \beta.$$

The fact that C happens to be invariant by \mathcal{F} implies that T is also invariant by \mathcal{F} in the sense mentioned above. Besides Stokes formula shows that T is, indeed, a closed current. Since T is clearly positive, we conclude that T is a closed positive current invariant by \mathcal{F} .

Next we shall consider a more geometric point of view to study closed currents invariant by a foliation \mathcal{F} as above. This point of view relies on the notion of *transverse invariant measure* which is essentially due to J. Plante. Since these transverse invariant measures are naturally defined for *regular foliations*, we assume for the time being that $\text{Sing}(\mathcal{F}) = \emptyset$.

Since M is compact and $\text{Sing}(\mathcal{F}) = \emptyset$, we can consider a finite covering $\{V_i\}$ of M by foliated charts $h_i : V_i \rightarrow \mathbb{D} \times \Sigma_i$ of \mathcal{F} , where \mathbb{D} stands for the unit disc of \mathbb{C} . The fact that the h_i 's define a foliated atlas implies that the change of coordinates $h_j \circ h_i^{-1}(x, y)$ has the special form $(f_{ij}(x, y), \gamma_{ij}(y))$.

Definition 2.3 *With the above notations, a transverse invariant measure for \mathcal{F} consists of a collection μ_i of (positive) finite measures over the transverse sections Σ_i which are invariant by change of coordinates. In other words, for every pair i, j and every Borel set $B \subset \Sigma_i$, one has $\mu_i(B) = \mu_j(\gamma_{ij}(B))$.*

Transverse invariant measures naturally provide closed (positive) currents invariant by the foliation \mathcal{F} in question by means of the following construction. Let $\{\phi_i\}$ be a partition of the unity subordinated to the finite covering $\{V_i\}$. Given a 2-form β , we define a current T by setting

$$T(\beta) = \sum_i \int_{\Sigma_i} \left(\int_{\text{Plaque}} (\phi_i \beta) \right) d\mu_i, \quad (1)$$

where the “Plaque” is naturally identified with the unit disc \mathbb{D} through the coordinates h_i 's. It is easy to check that T is a closed positive current. In fact, it is a continuous linear operator on the Fréchet space of smooth 2-forms. The conditions of being closed, positive and invariant by \mathcal{F} can immediately be checked. Conversely a closed positive current invariant by \mathcal{F} can be “desintegrated” to yield a transverse invariant measure so that the two objects turn out to be equivalent as originally pointed out by D. Sullivan [Su].

Let us now go back to the case of a holomorphic foliation \mathcal{F} with singularities. We then consider the open surface $M \setminus \text{Sing}(\mathcal{F})$ along with a covering $\{V_i\}$, $i \in \mathbb{N}$, by foliated charts $h_i : V_i \rightarrow \mathbb{D} \times \Sigma_i$ for the restriction of \mathcal{F} to $M \setminus \text{Sing}(\mathcal{F})$. Here the covering $\{V_i\}$ need not be finite. Again, if we are given a closed positive current T invariant by \mathcal{F} , the procedure of “desintegration” mentioned above can still be carried out word-by-word to yield a transverse invariant measure for the restriction of \mathcal{F} to $M \setminus \text{Sing}(\mathcal{F})$ as in Definition 2.3. Besides, from this transverse invariant measure we can recover the current T by means of Formula (1). Whereas

the “summation over i ” (the indices of foliated coordinates) may now be infinite, the series is naturally uniformly convergent so that it does define a current (i.e. a continuous operator) that actually coincides with T .

Here it might be worth making a minor comment concerning the passage from an “abstract” transverse invariant measure for \mathcal{F} to a closed positive current invariant by \mathcal{F} . This remark however will not be used anywhere in this work since we always start with a current already defined on M . Consider a transverse invariant measure for \mathcal{F} on the open set $M \setminus \text{Sing}(\mathcal{F})$ as in Definition 2.3 and the corresponding operator on smooth 2-forms given by (1). Since the summation over i is possibly infinite, it is necessary to make sure that the operator in question is well-defined and continuous. This clearly amounts to bound the mentioned integral on a neighborhood of the singular points of \mathcal{F} . Whereas I believe that this bound always exist in the context of holomorphic foliations on complex surfaces, this problem cannot *a priori* be reduced to an application of some Riemann extension or Hartogs theorem. The difficulty here being that we do not know *a priori* whether or not the corresponding integration of 2-forms is well-defined on a punctured neighborhood of the singularity in question. We shall not elaborate on this discussion since, as mentioned, it is not necessary for our purposes.

Let us finish this paragraph with a well-known lemma that will often be used in the course of this work.

Lemma 2.4 *Let T be a closed positive current invariant by \mathcal{F} . Assume that a point $p \in M \setminus \text{Sing}(\mathcal{F})$ has positive mass with respect to the transverse invariant measure for \mathcal{F} induced by T . Then the leaf L_p of \mathcal{F} through this point is contained in a compact curve.*

Proof. To prove the statement, let \overline{L}_p denote the closure of L_p . We just need to show that the set $\overline{L}_p \setminus L_p$ formed by the (proper) accumulation points of L_p is contained in the singular set of \mathcal{F} . Indeed, since $\text{Sing}(\mathcal{F})$ has codimension 2, it follows from the classical theorem of Remmert-Stein that \overline{L}_p is an analytic set.

To check the claim, suppose for a contradiction that q is a regular point of \mathcal{F} belonging to $\overline{L}_p \setminus L_p$. Consider a trivializing coordinate around q . Since $q \in \overline{L}_p \setminus L_p$, there exists a sequence of points $\{p_i\} \subset L_p$ such that $p_i \rightarrow q$. Besides for $i \neq j$, p_i, p_j belong to different plaques of the mentioned foliated chart. Denoting by Σ the corresponding local transversal, the measure on Σ associated to each of these plaques is a positive constant. It then follows from Equation 1 that the corresponding current has “infinite mass”, i.e. the integrals in (1) diverge for a suitable choice of β . The resulting contradiction establishes the lemma. \square

2.3 Brief review of Bonatti-Langevin-Moussu

In this paragraph we shall expand on the method developed in [B-L-M] to producing hyperbolic holonomy for certain holomorphic foliations (cf. also [G] and references therein). The study of the oriented foliation \mathcal{H} consisting of trajectories yielding “contractive holonomy” is the central object of this section.

Consider a surface M endowed with a holomorphic foliation \mathcal{F} as before. Let ω be a global non-closed meromorphic 1-form defining \mathcal{F} on M . The existence of this form is guaranteed if M is projective (cf. Section 2.1). Also denote by $(\omega)_0$ (resp. $(\omega)_\infty$) the divisor of zeros (resp. poles) of ω . Note that, in most applications, the sets $(\omega)_0, (\omega)_\infty$ are viewed as ordinary algebraic curves rather than as divisors (i.e. no multiplicity is associated to their components).

Next let ω_1 be the 1-form defined by

$$d\omega = \omega \wedge \omega_1. \quad (2)$$

To obtain a 1-form ω_1 satisfying the equation above it suffices to find a meromorphic vector field on M such that $\omega(X) = 1$. In fact, for this vector field X we have $d\omega = \omega \wedge \mathcal{L}_X(\omega)$, where \mathcal{L}_X stands for the Lie derivative. Note also that two 1-forms satisfying the mentioned equation must differ by a multiple of ω . In particular it follows that the values of ω_1 on vectors tangent to \mathcal{F} are well-defined even though ω_1 is not so. This ambiguity however can be avoided if ω_1 is regarded as a *foliated 1-form* (as opposed to an “ordinary” 1-form). By a foliated 1-form, it is meant a 1-form that is defined only for vectors tangent to (regular) leaves of \mathcal{F} . In other words, a foliated 1-form is not a usual 1-form on M since at a generic point of p this form is not defined for vectors in $T_p M$ that are transverse to the leaf of \mathcal{F} through p . Still another way of thinking of a foliated 1-form consists of saying that it is a meromorphic section of the cotangent bundle of \mathcal{F} , cf. Section 2.1. The preceding discussion can then be summarized by stating that Equation (2) unequivocally defines a meromorphic foliated 1-form on M . This foliated 1-form will systematically be denoted by ω_1 .

The foliated 1-form ω_1 can explicitly be computed. If (x, y) are local coordinates about a regular point p of \mathcal{F} in which $\omega = F(x, y)dy$ then ω_1 is given by

$$-\frac{\partial F / \partial x}{F} dx.$$

Clearly the above definition is compatible with foliated changes of coordinates so that it gives rise to a global (meromorphic) foliated 1-form ω_1 on M or, equivalently, to a global meromorphic section of the cotangent bundle of \mathcal{F} . This formula also shows that the form ω_1 is holomorphic on a neighborhood of p unless p belongs to the union of $(\omega)_0$ and $(\omega)_\infty$. A more accurate statement concerning the holomorphic nature of ω_1 is given below.

Lemma 2.5 *Let $p \in M$ be a regular point of \mathcal{F} . Suppose that all the irreducible components of $(\omega)_0 \cup (\omega)_\infty$ passing through p are invariant by \mathcal{F} . Then ω_1 is holomorphic at p .*

Proof. Consider foliated coordinates (x, y) about p so that ω becomes $F(x, y)dy$. We can assume that p belongs to $(\omega)_0 \cup (\omega)_\infty$, otherwise ω_1 is holomorphic as already seen. Nonetheless the assumption that all components of $(\omega)_0 \cup (\omega)_\infty$ passing through p are invariant by \mathcal{F} implies that there can be only one component which coincides in the coordinates (x, y) with the axis $\{y = 0\}$. In other words, we have $\omega = F(x, y)dy = y^k f(x, y)dy$ where $k \neq 0$ and for some holomorphic function f satisfying $f(0, 0) \neq 0$. Now a direct computation yields

$$\omega_1 = -\frac{\partial F / \partial x}{F} dx = -\frac{\partial f / \partial x}{f} dx.$$

The statement follows since $f(0, 0) \neq 0$. □

Conversely we have:

Lemma 2.6 *Let $C \subset M$ be an irreducible component of $(\omega)_0 \cup (\omega)_\infty$ that is not invariant by \mathcal{F} . Then ω_1 has poles of order 1 over C .*

Proof. Let $p \in C$ be a regular point for \mathcal{F} which does not belong to any irreducible component of $(\omega)_0 \cup (\omega)_\infty$ different from C itself. It suffices to show that the divisor of poles of ω_1 locally coincides with C with multiplicity equal to 1. As before we can choose foliated coordinates (x, y) about p where $\omega = F(x, y)dy = x^k f(x, y)dy$, $k \neq 0$, for some holomorphic function f satisfying $f(0, 0) \neq 0$. Now

$$\omega_1 = -\frac{\partial F/\partial x}{F}dx = -\frac{k}{x} - \frac{\partial f/\partial x}{f}dx.$$

The statement follows. \square

Consider now a regular leaf $L \subset M$ of \mathcal{F} where ω_1 does not vanish identically. The restriction of ω_1 to L is a meromorphic 1-form on the Riemann surface L (we shall often say that it is an abelian form on L). Therefore it induces a pair of (real one-dimensional) oriented singular foliations on L , namely the foliations given by $\{\text{Im}(\omega_1) = 0\}$ and $\{\text{Re}(\omega_1) = 0\}$. These foliations will respectively be denoted by \mathcal{H} and \mathcal{H}^\perp and they are mutually orthogonal for the underlying conformal structure of L . The orientation of \mathcal{H} (resp. \mathcal{H}^\perp) is determined by the increasing direction of $\text{Re}(\omega_1)$ (resp. $\text{Im}(\omega_1)$). More generally, the conformal structure of L also allows us to define the oriented foliation \mathcal{H}^θ whose trajectories form an angle θ with those of \mathcal{H} (where θ belongs to $(-\pi/2, \pi/2)$). Finally by letting the leaf L vary, the foliations $\mathcal{H}, \mathcal{H}^\perp$, or more generally \mathcal{H}^θ , can also be viewed as singular foliations defined on M . We shall return to this point when discussing the singularities of $\mathcal{H}, \mathcal{H}^\perp$.

Now the discussion in Lemma 2.6 yields the following lemma borrowed from [B-L-M].

Lemma 2.7 *Let $p \in M$ be a regular point of \mathcal{F} and denote by L the leaf through p . Let $C \subset M$ be an irreducible component of $(\omega)_0 \cup (\omega)_\infty$ that is not invariant by \mathcal{F} . Then we have:*

1. *$p \in C$ but p does not belong to $(\omega)_0$. Then p is a source for \mathcal{H} . Precisely there is a (complex one-dimensional) local coordinate X along L where the restriction of ω_1 to L becomes $\omega_1 = m dX/X$, $m \in \mathbb{N}^*$. In particular the leaves of \mathcal{H} are radial lines emanated from $p \in L$ (identified to $0 \in \mathbb{C}$).*
2. *$p \in C$ but p does not belong to $(\omega)_\infty$. Then p is a sink for \mathcal{H} . Precisely there is a (complex one-dimensional) local coordinate X along L where the restriction of ω_1 to L becomes $\omega_1 = -m dX/X$, $m \in \mathbb{N}^*$. In particular the leaves of \mathcal{H} are radial lines converging to $p \in L$ (identified to $0 \in \mathbb{C}$).*

Proof. Consider the first case. Since p is regular, we have $\omega = dY/f(X, Y)$ where $f(0, 0) = 0$ for suitable coordinates X, Y . The fact that $f(0, 0) = 0$ follows from the assumption $p \in (\omega)_\infty$ and $p \notin (\omega)_0$. Modulo performing a further change of coordinates, we can assume without loss of generality that $f(X, 0) = X^m$ for some $m \in \mathbb{N}^*$. Now the equation $d\omega = \omega \wedge \omega_1$ yields the desired form for ω_1 . The second case can analogously be treated. \square

Naturally an analogous discussion applies to the foliations \mathcal{H} (with $\theta \in (-\pi/2, \pi/2)$). Thus we already know that components of $(\omega)_0 \cup (\omega)_\infty$ that are not invariant by \mathcal{F} give rise to singularities of $\mathcal{H}, \mathcal{H}^\perp$ at regular points of \mathcal{F} . Next the foliated form ω_1 also have a divisor of zeros (resp. poles) denoted by $(\omega_1)_0$ (resp. $(\omega_1)_\infty$). It follows from the combination of Lemma 2.5 and Lemma 2.6 that $(\omega_1)_\infty \subset (\omega)_0 \cup (\omega)_\infty$. Besides no irreducible component of $(\omega_1)_\infty$ can be invariant by \mathcal{F} . Let us now consider a component C of $(\omega_1)_0$ that is not invariant by \mathcal{F} . The following lemma is also borrowed from [B-L-M].

Lemma 2.8 *Let p be a regular point of \mathcal{F} which does not belong to $(\omega)_0 \cup (\omega)_\infty$. Suppose that p lies in a component C of $(\omega_1)_0$ that is not invariant by \mathcal{F} and denote by L the leaf of \mathcal{F} containing*

p. Then the behavior of \mathcal{H} at *p* is that of a saddle with $2m$ separatrices. Precisely, in suitable coordinates X along L , the restriction of ω_1 to L becomes $\omega_1 = mX^{m-1}dX$ for $m \geq 2$.

Proof: Since *p* is regular and $p \notin (\omega)_0 \cup (\omega)_\infty$, there are local coordinates X, Y around *p* in which $\omega = f(X, Y)dY$ with f holomorphic. Suppose first that $f(X, 0)$ is not trivial. Then the restriction of ω_1 to $\{Y = 0\}$ is given by $-(\partial f / \partial X)dX/f$ where the functions are evaluated at $(X, 0)$. The result then follows. On the other hand, if ω vanishes identically on $\{Y = 0\}$ (or has poles over this leaf) then $\omega = y^k f Y$ with f as before. This still gives $\omega_1 = -(\partial f / \partial X)dX/f$ so that the statement follows. \square

Remark 2.9 Note that $mX^{m-1}dX$ is nothing but the lift of the regular form dX through the ramified covering $X \mapsto X^m$. In particular $mX^{m-1}dX$ has $2m$ separatrices (namely the lifts of the separatrices \mathbb{R}_+ and \mathbb{R}_- of dX) with alternate orientation.

Let us split the divisor of zeros $(\omega)_0$ of ω into two divisors $(\omega)_0^{\mathcal{F}}$ and $(\omega)_0^{\perp \mathcal{F}}$ as follows: an irreducible component C of $(\omega)_0$ belongs to $(\omega)_0^{\mathcal{F}}$ if and only if it is invariant by \mathcal{F} . Otherwise it belongs to $(\omega)_0^{\perp \mathcal{F}}$ (the multiplicity of each component remaining unchanged). Similarly we define the split of $(\omega)_\infty$ into $(\omega)_\infty^{\mathcal{F}}$ and $(\omega)_\infty^{\perp \mathcal{F}}$.

Let us now summarize the information so far obtained about possible singular points of \mathcal{H} , \mathcal{H}^\perp (and of \mathcal{H}^θ). Singular points for these foliation belong to the list below.

1. Singular points of \mathcal{F} (to be detailed later).
2. Irreducible components of $(\omega)_0^{\perp \mathcal{F}}$. These points are sink singularities for \mathcal{H} .
3. Irreducible components of $(\omega)_\infty^{\perp \mathcal{F}}$. These points are source singularities for \mathcal{H} .

Note that the foliated 1-form ω_1 is holomorphic away from $\text{Sing}(\mathcal{F}) \cup (\omega)_0^{\perp \mathcal{F}} \cup (\omega)_\infty^{\perp \mathcal{F}}$. In particular the support of the pole divisor of ω_1 are the union of the components of $(\omega)_0^{\perp \mathcal{F}}$ and $(\omega)_\infty^{\perp \mathcal{F}}$. Splitting also the zero divisor $(\omega_1)_0$ of ω_1 into divisors $(\omega_1)_0^{\mathcal{F}}$ and $(\omega_1)_0^{\perp \mathcal{F}}$, consisting respectively of components that are invariant by \mathcal{F} and components that are not invariant by \mathcal{F} , we identify the locus of the possible additional singularities of \mathcal{H} (resp. \mathcal{H}^\perp , \mathcal{H}^θ), namely:

4. Irreducible components of $(\omega_1)_0^{\perp \mathcal{F}}$. These components yield saddle singularities for \mathcal{H} (resp. \mathcal{H}^\perp , \mathcal{H}^θ).
5. Irreducible components of $(\omega_1)_0^{\mathcal{F}}$. Over these compact leaves of \mathcal{F} the foliations \mathcal{H} , \mathcal{H}^\perp and \mathcal{H}^θ are not defined.

We are now able to give the geometric meaning of the foliated form ω_1 . Consider a path c contained in a leaf L of \mathcal{F} along with local transverse sections Σ_0, Σ_1 respectively through $c(0), c(1)$. The parallel transport over the leaves of \mathcal{F} gives rise to a local diffeomorphism $\text{Hol}(c)$ from Σ_0 to Σ_1 taking $c(0)$ to $c(1)$ called the holonomy map of \mathcal{F} over c . It is well-known that the derivative of $\text{Hol}(c)$ is not intrinsically defined unless c is a loop. These derivatives however can be considered for a fixed choice of parametrizations for the transverse sections Σ_0, Σ_1 (and for fixed parametrizations they can be considered whether or not c is a loop). To be more precise,

suppose that Σ_0, Σ_1 are parameterized by ω , i.e. consider local coordinates $\varphi_i : \Sigma_i \rightarrow \mathbb{C}, i = 0, 1$, defined by

$$\varphi_0(p) = \int_{c(0)}^p \omega \quad \text{and} \quad \varphi_1(q) = \int_{c(1)}^q \omega$$

where the integrals are well-defined modulo choosing Σ_0, Σ_1 simply connected. In these coordinates, the holonomy map $\text{Hol}(c)$ can be identified to a local diffeomorphism of $(\mathbb{C}, 0)$. This local diffeomorphism satisfies

$$(\text{Hol}(c))'(c(0)) = \exp \left(- \int_c \omega_1 \right). \quad (3)$$

This formula is sometimes referred to as Poincaré Lemma. Several comments are needed here to fully explain its meaning. First let us fix a (finite) covering of a compact part K of $M \setminus \text{Sing}(\mathcal{F})$ by foliated coordinates $\varphi_i : U_i \rightarrow \mathbb{C}^2$ where each U_i is equipped with a local transverse section Σ_i parametrized by ω as above. Setting $\varphi_i(U_i) = D \times \mathbf{T}_i$ where D stands for the unit disc and where \mathbf{T}_i is identified with Σ_i parameterized as indicated, the Poincaré Lemma becomes applicable to every path $c \subset K$ contained in a leaf of \mathcal{F} (modulo an obvious decomposition of c into paths contained in the open sets U_i). Some further remarks are needed:

- The neighborhoods U_i are chosen so that $(\omega)_0^{\perp \mathcal{F}} \cap \varphi_i^{-1}(\partial D \times \mathbf{T}_i) = \emptyset$. Similarly $(\omega)_\infty^{\perp \mathcal{F}} \cap \varphi_i^{-1}(\partial D \times \mathbf{T}_i) = \emptyset$ and $(\omega_1)_0^{\perp \mathcal{F}} \cap \varphi_i^{-1}(\partial D \times \mathbf{T}_i) = \emptyset$.
- If C is a component of $(\omega)_0^{\mathcal{F}}$ (in particular C is invariant by \mathcal{F}), then the parametrization of Σ_i is actually ramified at the “origin” (it is a local ramified covering rather than a local diffeomorphism). An analogous conclusion (on a neighborhood of infinity) applies to components of $(\omega)_\infty^{\mathcal{F}}$.
- If $K' \subset K$ is a compact part of $M \setminus (\text{Sing}(\mathcal{F}) \cup (\omega)_0^{\mathcal{F}} \cup (\omega)_\infty^{\mathcal{F}})$ then the parametrization of Σ_i restricted to K' is “equivalent” to the parametrization induced by an auxiliary Hermitian metric on M (here it is to be noted that the first item above ensures that no Σ_i intersects $(\omega)_0^{\perp \mathcal{F}}, (\omega)_\infty^{\perp \mathcal{F}}$). In fact, every Hermitian metric on M induces parametrization that are pairwise “equivalent” in the sense that “corresponding lengths” are mutually controlled, from below and by above, by multiplicative constants.
- If $c \subset K'$ is a path contained in a trajectory l of \mathcal{H} , $l \subset L$ where L is a leaf of \mathcal{F} , then the holonomy $\text{Hol}(c)$ is such that $(\text{Hol}(c))'(c(0))$ is strictly smaller than 1 (with respect to above fixed foliated coordinates). Indeed, by construction, the integral of ω_1 over c increases monotonically with the length of c , cf. Formula (3).

Throughout the paper, we shall assume that \mathcal{F} is not a pencil, that is, not all the leaves of \mathcal{F} are compact. According to Jouanolou [J] this actually means that \mathcal{F} leaves only finitely many algebraic curves invariant. In particular the support of $(\omega)_0^{\mathcal{F}} \cup (\omega)_\infty^{\mathcal{F}} \cup (\omega_1)_0^{\mathcal{F}}$ consists of finitely many algebraic curves (if not empty).

We are now ready to explain the fundamental observation of [B-L-M]. Let K' be as above and consider a path $c \subset L \cap K'$ parametrizing a trajectory of \mathcal{H} (i.e. $\omega_1(c(t)).c'(t)$ is always a nonnegative real number). In particular the holonomy map $\text{Hol}(c)$ (measured with respect to the identifications fixed above) is such that $(\text{Hol}(c))'(c(0))$ decays exponentially with the *length of c* . The notion of *length of c* can be identified with the length measured in D for each coordinate φ_i . Alternatively this length can be measured with respect to the fixed auxiliary Hermitian metric on M (the two notions of lengths being equivalent up to multiplicative constants, i.e. the metrics

induced on L are quasi-isometric). Also, because c is contained in K' , the notions of distance in the transverse sections Σ_i induced by the parametrization through ω and by the mentioned Hermitian metric are mutually controlled by multiplicative constants.

Let then c be defined on the interval $[0, t_0] \subset \mathbb{R}$. The corresponding holonomy map $\text{Hol}(c)$ is then defined on a small disc $D_0(r) \subset \Sigma_{i_0}$ (for some i_0). Naturally $\text{Hol}(c)$ maps $D_0(r)$ diffeomorphically onto a neighborhood of $c(t_0) \in \Sigma_{i_1}$ (for some i_1). It is observed in [B-L-M] that the contractive character of the holonomy along the oriented leaves of \mathcal{H} allows one to have a uniform bound on the radius of $D_0(r)$ regardless of the point $c(t_0)$ and of the length of the c . Denoting by $\mathcal{H}_{|K'}$ the restriction of \mathcal{H} to K' one has:

Theorem 2.10 ([B-L-M]) *There is a uniform $r > 0$ with the following properties:*

1. *Let l_p be an oriented trajectory of $\mathcal{H}_{|K'}$ passing through $p \in K'$. If $c : [0, t_0] \rightarrow l_p \subset L_p \subset K$ is a parametrization of (a segment of) l_p ($p = c(0)$), then the corresponding holonomy map $\text{Hol}(c)$ is defined on $D_p(r) \subseteq \Sigma_{i_0}$ (for some i_0). Besides $\text{Hol}(c)$ maps $D_p(r)$ diffeomorphically onto its image in Σ_{i_1} (for some i_1).*
2. *Assume, in addition, that the distance of l_p to the divisor $(\omega_1)_0^{\perp \mathcal{F}}$ is bounded from below by a positive constant δ . Then there are uniform constant $C > 0$, $k > 0$ (k depending solely on δ) such that*

$$|(\text{Hol}(c))(q)| \leq C \exp(-k \text{length}(c)/2), \quad (4)$$

for every $q \in D_p(r)$ and where $\text{length}(c)$ stands for the length of the path c . □

In item 2 above, it is to be noted that the asymptotic exponential decay of the diameter of the set $(\text{Hol}(c))(D_p(r))$ has an intrinsic meaning since the length of c (as well as the the notion of distance in the transverse sections Σ_i restricted to K') vary in a way controlled by multiplicative constants as pointed out above. In particular if these metrics are changed, Formula (4) remains valid modulo changing the values of the constants C, k .

Remark 2.11 The reader will check that the same statement remains true for the foliations \mathcal{H}^θ for a fixed θ in $(-\pi/2, \pi/2)$. All these statements will be revisited and sharpened later in this paper.

3 The structure of \mathcal{H} around a singularity in the Siegel domain

The local structure of \mathcal{H} around a regular point of \mathcal{F} was described in the preceding section. The next step is to discuss the analogous problem on a neighborhood of a singularity p of \mathcal{F} which belongs to the Siegel domain. About this singularity there are coordinates (u, v) ($p \simeq (0, 0)$) in which ω becomes

$$\omega = h(u, v)[\lambda_1 u(1 + r^1(u, v)) dv + \lambda_2 v(1 + r^2(u, v)) du] \quad (5)$$

where h is meromorphic and r^1, r^2 are holomorphic functions verifying $r^1(0, 0) = r^2(0, 0) = 0$. Finally one also has $\lambda_1 \lambda_2 \neq 0$ and $\lambda_1 / \lambda_2 \in \mathbb{R}_+$ (as to the sign conventions, note that we are now using differential forms, rather than vector fields, to represent a singularity in the Siegel domain).

In particular \mathcal{F} possesses exactly 2 separatrices at $p \simeq (0, 0)$ namely, those given by $\{u = 0\}$ and $\{v = 0\}$. Throughout this section we work under the following extra-assumption:

Local invariance condition: One has $h(u, v) = u^a v^b$ for some $a, b \in \mathbb{Z}$.

The contents of the local invariance condition is that, on a small neighborhood of p , the curves $(\omega)_0$ and $(\omega)_\infty$ are invariant by \mathcal{F} . As it will be shown later, this assumption does not affect the generality of our arguments since it can always be obtained by performing finitely many blow-ups.

Because of the local invariance condition, the 1-form ω can be written in the coordinates u, v as

$$\omega = u^a v^b [\lambda_1 u(1 + r^1(u, v))dv + \lambda_2 v(1 + r^2(u, v))du]. \quad (6)$$

As seen in Section 2.3, the foliated form ω_1 can be obtained by restriction to the leaves of \mathcal{F} of an actual (locally defined) 1-form Ω_1 satisfying $d\omega = \omega \wedge \Omega_1$. Setting $\Omega_1 = f dv + g du$, it follows that

$$f \lambda_2 v(1 + r^2) - g \lambda_1 u(1 + r^1) = \lambda_1(1 + a)(1 + r^1) + \lambda_1 u r_u^1 - \lambda_2(1 + b)(1 + r^2) - \lambda_2 v r_v^2 \quad (7)$$

where r_u^1 (resp. r_v^2) stands for the partial derivative of r^1 (resp. r^2) with respect to u (resp. v).

Let us first consider the behavior of \mathcal{H} on the separatrices $\{u = 0\}$ and $\{v = 0\}$.

Lemma 3.1 *Suppose that $\lambda_1(1 + a) - \lambda_2(1 + b) \neq 0$. Then the behavior of \mathcal{H} over one of the separatrices is that of a sink (as in Lemma 2.7). Besides, on the other separatrix, \mathcal{H} behaves like a source.*

Proof: The restriction of Ω_1 to $\{v = 0\}$ is the Abelian form $g(u, 0)du$. By letting $v = 0$ in Equation (7), we obtain

$$-g(u, 0)\lambda_1 u(1 + r^1(u, 0)) = \lambda_1(1 + a)(1 + r^1(u, 0)) + \lambda_1 u r_u^1(u, 0) - \lambda_2(1 + b)(1 + r^2(u, 0)).$$

Hence

$$g(u, 0) = -\frac{\lambda_1(1 + a) - \lambda_2(1 + b)}{\lambda_1 u} + \tilde{s}_g(u) \quad (8)$$

where $\tilde{s}_g(u)$ is holomorphic around $0 \in \mathbb{C}$. Similarly, on $\{u = 0\}$, Ω_1 becomes $f(0, v)dv$ and Equation (7) yields

$$f(0, v) = \frac{\lambda_1(1 + a) - \lambda_2(1 + b)}{\lambda_2 v} + \tilde{s}_f(v) \quad (9)$$

where $\tilde{s}_f(v)$ is holomorphic around $0 \in \mathbb{C}$. Since $\lambda_1/\lambda_2 \in \mathbb{R}_+$, the statement follows from comparing Equations (8) and (9) and recalling that the restriction of Ω_1 to the leaves of \mathcal{F} coincides with ω_1 . \square

Remark 3.2 In the case where $\lambda_1(1 + a) - \lambda_2(1 + b) = 0$ the calculation above shows that ω_1 is holomorphic over both separatrices of \mathcal{F} at p .

As a matter of fact we also need to control ω_1 (or equivalently to understand the trajectories of \mathcal{H}) on the leaves of \mathcal{F} distinct from the separatrices. To abridge notations, in the sequel \mathcal{F} is going to be considered as a foliation defined on a neighborhood U of $(0, 0) \in \mathbb{C}^2$. The corresponding arguments should also be understood modulo reducing this neighborhood.

Recall that \mathcal{F} is defined on U by the 1-form

$$\eta = \lambda_1 u(1 + r^1)dv + \lambda_2 v(1 + r^2)du$$

so that $\omega = u^a v^b \eta$. For the rest of this section we always suppose that $\lambda_1(1 + a) - \lambda_2(1 + b) \neq 0$.

Lemma 3.3 *The form ω_1 has no zeros on U . In fact, there is a positive constant $C > 0$ such that for every $p \in L \subset U$ and unit vector $\mathbf{v} \in T_p L$ one has*

$$\|\omega_1(p) \cdot \mathbf{v}\| \geq C > 0.$$

Proof: Given $\epsilon_1, \epsilon_2 > 0$ sufficiently small, let us denote by Σ the local transverse section defined by

$$\Sigma = \{(u, v) \in \mathbb{C}^2 ; u = \epsilon_1 \text{ and } |v| < \epsilon_2\}.$$

Denoting by $\Sigma_{\mathcal{F}}$ the saturated of Σ by \mathcal{F} , it is proved in [M-M] (cf. also [Ma], [Re]) that $\Sigma_{\mathcal{F}} \cup \{u = 0\}$ contains a neighborhood of $(0, 0) \in \mathbb{C}^2$. Since we need a parametrization of the leaf L in order to estimate the restriction of ω_1 to L , let us consider the set $D_1^- = \{(u, v) \in \mathbb{C}^2 ; |u| < \epsilon_1 \text{ and } v \notin \mathbb{R}_-\}$. We then define $W = \{(u, v) \in \mathbb{C}^2 ; u \in D_1^- \text{ and } |v| \leq \epsilon_2\}$. Because D_1^- is simply connected, the restriction of \mathcal{F} to W does not present the local holonomy associated to the separatrix $\{v = 0\}$. Thus fixed $y_0 \in \Sigma$, the leaf L_0 of \mathcal{F} restricted to W through y_0 is the graph of a holomorphic function h . Precisely the argument of [M-M] shows the existence of $h : D_{y_0} \subset D_1^- \rightarrow \mathbb{C}$ whose graph $\{(u, h(u))\}$, $u \in D_{y_0}$, coincides with L_0 .

Clearly to obtain estimates for the restriction of ω_1 to the leaves of \mathcal{F} , it suffices to estimate ω_1 over leaves L_0 as above since \mathbb{R}_- can be substituted by another semi-line in the definition of D_1^- . Now, when dealing with L_0 , we are allowed to use the parametrization $u \mapsto (u, h(u))$. Fix a point $q = (u_q, h(u_q)) \in L_0$. The tangent space to L_0 at q is spanned over \mathbb{C} by the vector $(1, h'(u_q))$ whose norm is not uniformly bounded on U . In any case $\omega_1(q)$ evaluated over $(1, h'(u_q))$ coincides with the evaluation of Ω_1 over the same vector. Thus we obtain

$$\omega_1(q) \cdot [1, h'(u_q)] = \Omega_1(q) \cdot [1, h'(u_q)] = f \cdot h'(u_q) + g. \quad (10)$$

On the other hand, $\omega(q) \cdot [1, h'(u_q)] = 0$ so that Formula (7) provides

$$h'(u_q)[1 + r^1(u_q, h(u_q))] = \frac{\lambda_2 h(u_q)}{\lambda_1 u_q} (1 + r^2(u_q, h(u_q))). \quad (11)$$

Therefore

$$\begin{aligned} f \cdot h'(u_q) + g &= -f \frac{\lambda_2 h(u_q)}{\lambda_1 u_q} \frac{1 + r^2}{1 + r^1} + g \\ &= \frac{-1}{\lambda_1 u_q (1 + r^1)} [\lambda_1 (1 + a)(1 + r^1) + \lambda_1 u_q r_u^1 - \lambda_2 (1 + b)(1 + r^2) - \lambda_2 h(u_q) r_v^2] \end{aligned} \quad (12)$$

where the functions r^1, r^2, r_u^1, r_v^2 are evaluated at $(u_q, h(u_q))$, cf. Formula (7). Now recall that $\|u_q\|$ and $\|h(u_q)\|$ are bounded by ϵ_1, ϵ_2 . It follows from (11) that the norm of $(1, h'(u_q))$ is bounded by $\max\{1, \text{const}/\|u_q\|\}$ for a suitable constant const . The statement then results from the condition $\lambda_1(1 + a) - \lambda_2(1 + b) \neq 0$. \square

We still need to describe the geometry of the leaves of \mathcal{H} on L_0 . According to Lemma (3.1), we can suppose without loss of generality that the oriented leaves of \mathcal{H} on $\{v = 0\}$ converge to

$0 \in \{v = 0\} \subset \mathbb{C}^2$ (i.e. $0 \simeq (0, 0)$ is a sink for \mathcal{H} over $\{v = 0\}$). Similarly $0 \in \{u = 0\} \subset \mathbb{C}^2$ is a source for the leaves of \mathcal{H} contained in $\{u = 0\}$. Next we consider the (real 3-dimensional) set

$$A = \{(u, v) \in \mathbb{C}^2 ; |u| = \epsilon_1 \text{ and } |v| < \epsilon_2\}.$$

If ϵ_1, ϵ_2 are appropriately chosen and sufficiently small, the the oriented leaves of \mathcal{H} point inwards A , i.e. at a point $(u, v) \in A$ the leaf of \mathcal{H} through this point is oriented in the decreasing direction of the absolute value of u .

Now let l be an oriented leaf of the restriction of \mathcal{H} to a small neighborhood of $(0, 0) \in \mathbb{C}^2$. As it will shortly be seen, l is not closed. If $q_1, q_2 \in l$, we say that $q_2 > q_1$ provided that one can move from q_1 to q_2 in the sense of the orientation of l . We also denote by $\text{dist}(q_1, q_2)$ the length of the segment of l whose extremities are q_1, q_2 . Finally we are ready to state the main result of this section.

Proposition 3.4 *There is a neighborhood V of $(0, 0) \in \mathbb{C}^2$ with the following properties:*

1. *Given $q_1 \in l \cap V$, there is $q_2 \in l \cap A$, with $q_1 > q_2$ and such that $\text{dist}(q_1, q_2) < \text{const } \epsilon_1$.*
2. *Given $q_1 \in l \cap V$, there is $\bar{q} = (\bar{q}^1, \bar{q}^2) \in l$, with $\bar{q} > q_1$ and $\text{dist}(q_1, \bar{q}) < \text{const } \epsilon_2$. Besides $|\bar{q}^2| = \epsilon_2$ and $\bar{q}^1 \in \pi_1(V)$ where $\pi_1(V)$ stands for the projection of V on the first coordinate.*

Proof: Let $B(\delta)$ be the bidisc $\{(u, v) \in \mathbb{C}^2 ; |u| < \delta \text{ and } |v| < \delta\}$. We are going to show that $B(\delta)$ satisfies the conditions in our statement provided that δ is sufficiently small. Consider $a \in B(\delta)$ and suppose without loss of generality that the real part $\text{Re}(q_1)$ of q_1 is positive. Let then L (resp. l) be the leaf of the restriction of \mathcal{F} (resp. trajectory of the restriction of \mathcal{H}) to $B(\delta)$ containing q_1 . As already seen, L is the graph of a holomorphic function $h : D_q \subset D_1^- \rightarrow \mathbb{C}$. In the parametrization $u \mapsto (u, h(u))$, the restriction of ω_1 to L becomes

$$fh' + g = \frac{\lambda_1(1+a) - \lambda_2(1+b) + \alpha}{\lambda_1 u} + s(u) \quad (14)$$

where s is holomorphic and α can be made arbitrarily small by reducing ϵ_1, ϵ_2 . Indeed Formula (14) is an immediate reformulation of Formula (13). In particular, one has $\lambda_1(1+a) - \lambda_2(1+b) + \alpha \neq 0$. We now set $q_1 = (u_1, h(u_1))$. Recalling that $D_q \subset \mathbb{C}$, we denote by R_q the radial line emanated from $0 \in \mathbb{C}$ and passing through u_1 . The intersection of R_q with the circle $|u| = \epsilon_1$ is $\epsilon_1 u_1 / |u_1|$. Similarly let $\pi_1(l)$ be the oriented leaf of $\{\text{Im}(fh' + g) = 0\}$ containing u_1 which is nothing but the projection of l on the first coordinate.

Claim: There is a point $u_2 \in \pi_1(l)$ such that $|u_2| = \epsilon_1$. Besides there is a uniform constant C such that

$$\text{dist} \left(u_2, \frac{\epsilon_1 u_1}{|u_1|} \right) < C \epsilon_1^2.$$

Proof of the Claim: It is an elementary fact about continuous/differentiable dependence of the initial conditions for solutions of real ordinary differential equations. The foliation associated to $\{\text{Im}[(\lambda_1(1+a) - \lambda_2(1+b))/\lambda_1 u] = 0\}$ consists of radial lines through $0 \in \mathbb{C}$ so that the assertion is trivial in this case. Nonetheless the foliation in which we are interested is given by an Abelian form whose distance to $(\lambda_1(1+a) - \lambda_2(1+b))/\lambda_1 u$ is less than $C \epsilon_1$ for an appropriate constant C . The statement promptly follows. \square

Combining the above claim with the fact that $\sigma_{\mathcal{F}} \cup \{u = 0\}$ contains a neighborhood of $(0, 0) \in \mathbb{C}^2$, we conclude that l intersects A at a point q_2 . Estimates in [M-M] (see also [Ma] and

[Re]) guarantee that q_2 satisfies the conditions in the statement. Analogously one proves that the continuation of l intersects the set $|v| = \epsilon_2$ at a point \bar{q} with the desired properties. For further details on these estimates we refer the reader to the quoted papers. \square

Corollary 3.5 *Under the preceding conditions the trajectories of \mathcal{H}^\perp contained in the local separatrices of \mathcal{F} are closed curves encircling the origin. For $\theta \in (-\pi/2, \pi/2)$, the trajectories of \mathcal{H}^θ contained in the local separatrix of \mathcal{F} where \mathcal{H} has a sink singularity (resp. a source singularity) are spiraling curves converging to the origin (resp. being emanated from the origin).*

Furthermore, on a local leaf of \mathcal{F} different from its separatrices, the behavior of \mathcal{H}^\perp is essentially determined by the local holonomy of the separatrices whereas the behavior of \mathcal{H}^θ , $\theta \in (-\pi/2, \pi/2)$, is the combination of the above described Dulac transform (cf. below) with a finite power of the mentioned local holonomy map.

Proof: It follows immediately from the fact that the oriented trajectories of \mathcal{H} (resp. \mathcal{H}^θ) form an angle of $\pi/2$ (resp. θ) with the oriented trajectories of \mathcal{H} . \square

Let us close this section with a discussion of the so-called *Dulac transform* associated to a singularity in the Siegel domain. Although this is a local discussion formally independent of the structure of \mathcal{H} , it naturally involves definitions and results discussed above so that here seems to be a good place to carry it out. The material below will also be used in Sections 4 and 6. Whereas classical in nature, it is not easy to find a detailed exposition of this material in the literature. First we resume some notations.

Recall that \mathcal{F} is defined on a neighborhood of $(0, 0) \in \mathbb{C}^2$ by the vector field

$$Y = \lambda_1 u(1 + r^1) \frac{\partial}{\partial u} - \lambda_2 v(1 + r^2) \frac{\partial}{\partial v} . \quad (15)$$

Recall also that $A \subset \mathbb{C}^2$ was defined as $A = \{(u, v) \in \mathbb{C}^2 ; |u| = \epsilon_1 \text{ and } |v| < \epsilon_2\}$. Similarly we set $B = \{(u, v) \in \mathbb{C}^2 ; |v| = \epsilon'_2 \text{ and } |u| < \epsilon'_1\}$ for certain $\epsilon'_1, \epsilon'_2 > 0$. Fixed u_0 with $|u_0| = \epsilon_1$ (resp. v_1 with $|v_1| = \epsilon'_2$), we denote by Σ_0^A (resp. Σ_1^B) the set $\{(u, v) \in \mathbb{C}^2 ; u = u_0 \text{ and } |v| < \epsilon_2\}$ (resp. $\{(u, v) \in \mathbb{C}^2 ; |u| < \epsilon'_1 \text{ and } v = v_1\}$). In the sequel $\epsilon_1, \epsilon'_1, \epsilon'_2$ are fixed and small whereas ϵ_2 can be made smaller whenever necessary.

For u_0, v_1 as above, let us denote by $\mathcal{F}_0^A, \mathcal{F}_1^B$ the saturated of Σ_0^A, Σ_1^B by \mathcal{F} . As already seen, both $\mathcal{F}_0^A \cup \{u = 0\} \cup \{v = 0\}$ and $\mathcal{F}_1^B \cup \{u = 0\} \cup \{v = 0\}$ contain an open neighborhood of $(0, 0) \in \mathbb{C}^2$. Therefore, up to choosing ϵ_2 very small, for every $(u_0, v_0) \in \Sigma_0^A$, there exist paths $c : [0, 1] \rightarrow L_{(u_0, v_0)}$ such that $c(0) = (u_0, v_0)$ and $c(1) \in \Sigma_1^B$ (where $L_{(u_0, v_0)}$ stands for the leaf of \mathcal{F} through (u_0, v_0)). If c, c' are two paths as above and satisfying $c(1) = (u, v_1)$, $c'(1) = (u', v_1)$, then u, u' belong to the same orbit of the local holonomy of the axis $\{u = 0\}$.

Now consider a simply connected domain $V_0 \subset \Sigma_0^A \setminus \{(u_0, 0)\}$. Suppose we are given a point $(u_0, v_0) \in V_0$ and a path $c_0 : [0, 1] \rightarrow L_{(u_0, v_0)}$ as before. For (u, v) sufficiently close to (u_0, v_0) , it is then possible to choose by continuity a path $c : [0, 1] \rightarrow L_{(u_0, v_0)}$ such that $c(0) = (u_0, v)$ and $c(1) \in \Sigma_1^B$. Since V_0 is simply connected, we can extend this definition to the whole of V_0 . In this way, we obtain a holomorphic map $\text{Dul} : V_0 \subset \Sigma_0^A \setminus \{(u_0, 0)\}$ to Σ_1^B . This map is going to be called the *Dulac transform* (which depends on the previously chosen path c_0). Identifying Σ_0^A with a neighborhood of $0 \in \mathbb{C}$, we shall refer to a sector of angle θ and radius r meaning the intersection of the ball of radius r with a sector of angle θ (and vertex at $0 \in \mathbb{C}$). In practice, V_0 will always be a sector of angle less than 2π and sufficiently small radius. The choice of the initial path c_0 and of the semi-line in question entirely determines the corresponding map Dul .

The following lemma consists again of estimates that can be found for example in [M-M], [Ma] or in [Re].

Lemma 3.6 *Let $V_0 = \Sigma_0^A$ be a sector of angle less than 2π and sufficiently small radius. Fix a path c and consider the resulting Dulac transform $\text{Dul} : V_0 \subset \Sigma_0^A \rightarrow \Sigma_1^B$. Then the following estimate holds*

$$\|\text{Dul}(v)\| \leq \text{Const} \|v\|^{\lambda_1/\lambda_2} (1 + O(\|v\|)) .$$

□

In particular, if $\lambda_1 > \lambda_2$, the behavior of Dul is that of a (strong) contraction provided that $\|v\|$ is small. When $\lambda_1 < \lambda_2$ then Dul behaves as an expansion for $\|v\|$ small.

Finally suppose that Σ_0^A, Σ_1^B are endowed with measures μ_0, μ_1 which are part of a system (of transverse sections and measures) defining a transverse invariant measure for a global realization of \mathcal{F} on some complex surface (in the sense of Section 2.2). Note that, in general, Dul is not one-to-one on $V_0 \subset \Sigma_0^A$ (if the angle of V_0 is not small) so that $\mu_0(V_0) \neq \mu_1(\text{Dul}(V_0))$. Nonetheless we have:

Lemma 3.7 *With the preceding notations the following is verified.*

1. *Suppose that $\lambda_1 > \lambda_2$ and let V_0 be a sector of angle slightly less than $2\pi\lambda_2/\lambda_1$. Then, for $\|v\|$ very small, Dul is one-to-one on V_0 and satisfies $\mu_0(V_0) = \mu_1(\text{Dul}(V_0))$.*
2. *Suppose that $\lambda_1 < \lambda_2$ and let V_0 be a sector of angle slightly less than 2π . Then, for $\|v\|$ very small, Dul is one-to-one on V_0 and satisfies $\mu_0(V_0) = \mu_1(\text{Dul}(V_0))$.*

Proof: The proof consists of showing that for $v \in W$, we can obtain flow boxes containing the corresponding paths $c : [0, 1] \rightarrow L_{(u_0, v)}$ so that the holonomy associated to these paths is well-defined and injective. This is clear when \mathcal{F} is linearizable. In the general case it results again from the asymptotic estimates already mentioned above. □

Remark 3.8 **The case when the restriction of ω_1 to the local separatrices is holomorphic:** the reader has noted that the discussion of the behavior of \mathcal{H} (resp. \mathcal{H}^\perp and \mathcal{H}^θ) carried out in Proposition 3.4 was based on the local invariance condition and on the assumption that $\lambda_1(1+a) - \lambda_2(1+b) \neq 0$. Now that we have already introduced the notion of Dulac transform associated to a Siegel singularity, let us also consider the case where $\lambda_1(1+a) - \lambda_2(1+b) = 0$ (assuming that the local invariance condition is still satisfied). As mentioned this case is such that the restriction of ω_1 to the invariant axes $\{y = 0\}$ and $\{x = 0\}$ is holomorphic. Thus the restriction of ω_1 to $\{y = 0\}$ (resp. $\{x = 0\}$) either is regular or vanishes at the origin. For the time being we shall assume that this restriction is not identically zero, though this is not strictly necessary for what follows (cf. Sections 4 and 5). Consider then the behavior of \mathcal{H} restricted to $\{y = 0\}$ and suppose there is a trajectory l of \mathcal{H} that passes “very close” to the origin. The first remark to be made here is that l can be “deformed” to avoid a fixed neighborhood of the origin. These deformations are similar to deformations already performed when a singularity converges to a saddle-singularity of \mathcal{H} occurring at a regular point of \mathcal{F} , cf. Section 2 and/or [B-L-M]. In particular they can be done without destroying the “contractive behavior” of the holonomy of \mathcal{F} associated to the trajectories of \mathcal{H} . Therefore, if needed, a Siegel singularity satisfying the condition $\lambda_1(1+a) - \lambda_2(1+b) = 0$ can be avoided by the trajectories of \mathcal{H} . In other words, the singularity becomes “invisible” and thus it can be ignored.

However, even if these singularities can be avoided, we might want to take advantage of them by exploiting the (local) saddle-behavior of \mathcal{F} . In other words, it may be useful to let a \mathcal{H} -trajectory to approximate the singularity so as to be continued “through the other separatrix of \mathcal{F} ”, i.e. the \mathcal{H} trajectory may go through the Dulac transform and then be continued in a different way. In this paper, if a trajectory of \mathcal{H} is about to entering some (previously fixed) neighborhood of a Siegel singularity as above, we shall consider all possible continuations of it, namely those that actually “avoid the singularity” and those that passes through the Dulac transform associated to the singularity itself. We shall return to these cases later in Sections 5 and 6.

An alternative point of view: let us close this section by explaining an alternate way to see the above results on Dulac transforms and their connections with the material of Section 2.3.

To begin with, let us make a simple remark concerning how the Dulac transform can be viewed in most of our applications. With the preceding notations suppose that the orientation of the trajectories of \mathcal{H} is such that the origin is a sink for the restriction of \mathcal{H} to $\{v = 0\}$. Then ϵ_1, ϵ_2 can be chosen so that \mathcal{H} is transverse to $A \subset \mathbb{C}^2$. Besides every \mathcal{H} -trajectory intersecting A points inward A and, unless this intersection occurs at a point belonging to $\{v = 0\}$, it will eventually intersect B with outward orientation. Thus we can define the Dulac transform as being the map from A to B defined by the trajectories of \mathcal{H} . Note that this map is locally holomorphic away from $A \setminus \{v = 0\}$. Besides, for $(u_0, v_0) \in A$, $v_0 \neq 0$, its image satisfy the estimates given in Lemma 3.6. Furthermore it is not hard to adapt the contents of Lemma 3.7 to this setting.

Naturally the preceding statements about the contractive or expansive character of the Dulac map can also be viewed in terms of Poincaré Lemma discussed in Section 2.3. For this it is however necessary to work with (possibly) ramified coordinates. Let us then consider a foliation \mathcal{F} defined on a neighborhood of $(0, 0) \in \mathbb{C}^2$ by the vector field Y in (15). More precisely suppose that the 1-form ω defining \mathcal{F} is actually $\omega = \lambda_1 u(1 + r^1) dv + \lambda_2 v(1 + r^2) du$. Consider also sections Σ_0^A and Σ_1^B as above and suppose that the orientation of the trajectories of \mathcal{H} is such that they go from Σ_0^A to Σ_1^B (i.e. $\lambda_1 > \lambda_2$). To apply Formula (3) to this case, we need to consider the parametrizations of Σ_0^A, Σ_1^B that are obtained through the integral of ω . It is then natural to set a coordinate z_1 on Σ_0^A and a coordinate z_2 on Σ_1^B such that

$$z_1 = \lambda_2 v(1 + \text{h.o.t.}) \quad \text{and} \quad z_2 = \lambda_1 u(1 + \text{h.o.t.}).$$

In these coordinates the derivative of the above introduced Dulac transform can be estimate by means of Formula (3). This amounts to estimating the integral of ω_1 over a segment of trajectory of \mathcal{H} going from Σ_0^A to Σ_1^B . The latter estimate however is essentially equivalent to the calculations performed above.

4 Singularities of \mathcal{F} and invariant measures

Now we are going to begin the analysis of the global setting where \mathcal{F} is a singular holomorphic foliation defined on a complex surface M . Throughout this section \mathcal{F} is supposed to admit an invariant positive closed current T whose associated transverse measure does not give mass to points (T is said to be diffuse). Let $\text{supp}(T) \subseteq M$ be the *support* of T which is obviously a compact set invariant by \mathcal{F} .

Modulo applying Seidenberg’s theorem, we can suppose that all the singularities of \mathcal{F} are reduced. Our first aim in this section is to establish Proposition (4.1) below.

Proposition 4.1 *Let $p \in \text{Sing}(\mathcal{F})$ be a singularity of \mathcal{F} lying in $\text{supp}(T)$. Then p is a singularity in the Siegel domain or it is an irrational focus. Furthermore if p belongs to the Siegel domain and has eigenvalues with rational quotient, then \mathcal{F} is linearizable around p .*

Since $p \in \text{Sing}(\mathcal{F}) \cap \text{supp}(T)$ is reduced, the proof of Proposition (4.1) essentially consists of showing that p is neither a hyperbolic singularity nor a saddle-node. These are the contents of Lemmas (4.2) and (4.3) below.

Lemma 4.2 *If $p \in \text{Sing}(\mathcal{F}) \cap \text{supp}(T)$, then p is not hyperbolic.*

Proof: Suppose for a contradiction that p is hyperbolic. Then Poincaré Theorem ensures that \mathcal{F} is linearizable around p . In other words, there are local coordinates u, v in which \mathcal{F} is given by

$$\eta = \lambda_1 u dv - \lambda_2 v du$$

with $\lambda_1/\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$. Consider a local transverse section Σ passing through the point $(1, 0)$. This section allows us to identify the local holonomy of the separatrix $\{v = 0\}$ with a local diffeomorphism h fixing $0 \in \mathbb{C}$. The condition $\lambda_1/\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$ implies that h is hyperbolic, i.e. $|h'(0)| < 1$. Now consider a (local) leaf L of \mathcal{F} contained in $\text{supp}(T)$ and intersecting Σ at a point $(1, z_0)$. Denote by μ_Σ a representative of T , viewed as transverse invariant measure, over Σ (cf. Section 2.2). If $z_0 \neq 0$, the orbit of $(1, z_0)$ under h consists of infinitely many points converging towards $(1, 0) \in \Sigma$. Furthermore, if $V \subset \Sigma$ is a sufficiently small neighborhood of $z_0 \simeq (1, z_0) \in \Sigma$, then the open sets $V, h(V), h^2(V), \dots$ are pairwise disjoint. Nonetheless they have all the same μ_Σ measure for h preserves μ_Σ . In addition $\mu_\Sigma(V) > 0$ since L is contained in $\text{supp}(T)$. Together these facts imply that $\mu_\Sigma(\Sigma) = \infty$ what is impossible. We then conclude that $\text{supp}(T)$ is locally contained in the separatrices of \mathcal{F} at p . Therefore μ_Σ has an atomic component which is necessarily concentrated over an algebraic curve. Since this is impossible, the lemma follows. \square

Through a similar argument we are going to prove that $p \in \text{Sing}(\mathcal{F}) \cap \text{supp}(T)$ cannot be a saddle-node either. A very complete reference for saddle-node singularities is [M-R]. The facts used below are however well-known. If \mathcal{F} is a saddle-node singularity then it can be written in Dulac Normal Form, i.e. in suitable local coordinates u, v , the foliation \mathcal{F} is given by the 1-form η satisfying

$$\eta = [u(1 + \Lambda v^p) + R(u, v)]dv - v^{p+1}du \quad \text{with} \quad \Lambda \in \mathbb{C} \quad \text{and} \quad p \geq 1.$$

In particular $\{v = 0\}$ is a separatrix of \mathcal{F} called the *strong invariant manifold* of \mathcal{F} . Considering a local transverse Σ as in Lemma (4.2), we can identify the holonomy of the strong invariant manifold to a (local) diffeomorphism h fixing $0 \in \mathbb{C}$. However, this time, h has the form $h(z) = z + z^{p+1} + \text{h.o.t.}$, where as usual h.o.t stands for terms of higher order.

Lemma 4.3 *If $p \in \text{Sing}(\mathcal{F}) \cap \text{supp}(T)$, then p cannot be a saddle-node.*

Proof: Consider \mathcal{F}, Σ and η as above. Other than the strong invariant manifold, a saddle-node may or may not possess another separatrix (necessarily smooth and transverse to the former one) which is called the weak invariant manifold. In particular a saddle-node possesses at least one and at most two separatrices.

We now suppose that $\text{supp}(T)$ is not locally contained in the union of the separatrices of \mathcal{F} since this would again lead us to a contradiction. It follows from [M-R] that the union of the saturated \mathcal{F}_Σ of Σ by \mathcal{F} with the weak invariant manifold (if it exists) contains a neighborhood of p . Thus there is a leaf $L \subset \text{supp}(T)$ of \mathcal{F} intersecting Σ at a point $(1, z_0)$ with $z_0 \neq 0$. The topological description of the dynamics of $h(z) = z + z^{p+1} + \text{h.o.t.}$ is well-known (cf. for example [C-G]) and it follows the existence of a small neighborhood $V \subset \Sigma$ of $z_0 \simeq (1, z_0)$ such that $V, h(V), h^2(V) \dots$ are pairwise disjoint. By taking a representative μ_Σ of T on Σ as in Lemma (4.2) we conclude that $\mu_\Sigma(\Sigma) = \infty$. This is however impossible and establishes the lemma. \square

Proof of Proposition (4.1): After Lemmas (4.2) and (4.3), we only need to prove that a Siegel singularity with rational eigenvalues is linearizable. As already seen \mathcal{F} is locally given by

$$\eta = \lambda_1 u(1 + \text{h.o.t.}) dv + \lambda_2 v(1 + \text{h.o.t.}) du$$

with $\lambda_1/\lambda_2 \in \mathbb{Q}_+$. Denoting by Σ a transverse section passing through $(1, 0)$, it was seen that the union of $\{u = 0\}$ with \mathcal{F}_Σ (the saturated of Σ by \mathcal{F}) contains a neighborhood of p . Thus, as before, there is a leaf $L \subset \mathcal{K}$ intersecting Σ at a point $(1, z_0)$. Without loss of generality we can suppose that $z_0 \neq 0$.

On the other hand, the linear part of the holonomy diffeomorphism h associated to $\{v = 0\}$ is precisely $e^{2\pi i \lambda_1/\lambda_2} z$. Thus a power of h is tangent to the identity. According to a result of Mattei-Moussu [M-M], \mathcal{F} is locally linearizable if and only if the power of h in question coincides with the identity. Hence we suppose for a contradiction that this power is tangent to the identity and different from the identity. In this case, however, it has the form $z + cz^k + \dots$ with $c \neq 0$. The final contradiction is then obtained as at the end of Lemma (4.3). The proposition is proved. \square

Summarizing the preceding discussion we can suppose that the (reduced) singularities of \mathcal{F} lying in $\text{supp}(T)$ are of one of the following types:

- a singularity in the Siegel domain.
- an irrational focus.

Note also that Poincaré Theorem still implies that an irrational focus is automatically linearizable. Hence, in this case, \mathcal{F} is locally given by the form

$$\eta = \lambda_1 u dv - \lambda_2 v du$$

with $\lambda_1/\lambda_2 \in \mathbb{R}_+ \setminus \mathbb{Q}_+$. It is easy to work out the structure of the foliation \mathcal{H} near to an irrational focus singularity. This is similar to the discussion carried out in Section 3 whereas technically simpler since \mathcal{F} is always linearizable. Again on a small neighborhood of p the curves $(\omega)_0$ and $(\omega)_\infty$ are supposed to be invariant by \mathcal{F} (local invariance condition). This means that ω can be written in local coordinates u, v as

$$\omega = h(u, v) u^a v^b [\lambda_1 u dv - \lambda_2 v du] \quad (16)$$

where $h(0, 0) \neq 0$. Setting $\Omega_1 = f dv + g du$ the equation $d\omega = \omega \wedge \Omega_1$ yields

$$h(u, v)(\lambda_2 v f + \lambda_1 u g) = -h(u, v)(\lambda_1(a+1) + \lambda_2(b+1)) - u \frac{\partial h}{\partial u} - v \frac{\partial h}{\partial v}. \quad (17)$$

In the sequel we suppose that a, b are not simultaneously equal to -1 so that $\lambda_1(a+1) + \lambda_2(b+1) \neq 0$ (recall that $\lambda_1/\lambda_2 \in \mathbb{R}_+ \setminus \mathbb{Q}_+$). By setting $u = 0$ (resp. $v = 0$) we conclude that the behavior

of h over the separatrix $\{u = 0\}$ (resp. $\{v = 0\}$) is either that of a sink or that of a source according to whether $\lambda_1(a + 1) + \lambda_2(b + 1) > 0$ or $\lambda_1(a + 1) + \lambda_2(b + 1) < 0$.

For the leaves of \mathcal{F} different from the separatrices, we can perform a discussion similar to the one carried out in Section 3 by exploiting the presence of the “multi-valued” first integral $u^{\lambda_2}v^{\lambda_1}$. The reader will easily check that the behavior of \mathcal{H} over the separatrices is repeated over the general leaves. The result is then summarized by

Proposition 4.4 *Let $p \in \text{Sing}(\mathcal{F}) \cap \text{supp}(T)$ be an irrational focus. Consider also local coordinates u, v defined on a bidisc of radius ϵ about p and suppose that ω is given by (16) where a, b are not simultaneously equal to -1 . If L is a leaf of \mathcal{F} , then the restriction of \mathcal{H} to L consists of lines of length less than $\text{Const} \cdot \epsilon$ for an appropriate uniform constant Const . Furthermore these lines converge to $(0, 0)$ if $\lambda_1(a + 1) + \lambda_2(b + 1) > 0$ (i.e. the end of the leaf corresponding to $(0, 0)$ is a sink). Similarly these lines are emanated from $(0, 0)$ if $\lambda_1(a + 1) + \lambda_2(b + 1) < 0$ (i.e. the end of the leaf corresponding to $(0, 0)$ is a source). \square*

Remark 4.5 An irrational focus $p \in \text{Sing}(\mathcal{F}) \cap \text{supp}(T)$ is going to be called a sink (resp. a source) if, with the notations of the lemma above, one has $\lambda_1(a + 1) + \lambda_2(b + 1) > 0$ (resp. $\lambda_1(a + 1) + \lambda_2(b + 1) < 0$). Sometimes we shall use the expressions sink-irrational focus or source-irrational focus to emphasize that we are dealing with an irrational focus singularity. This terminology also serves to distinguish between singularities of \mathcal{F} behaving as sinks (or sources) for \mathcal{H} and sinks (or sources) of \mathcal{H} occurring at regular points of \mathcal{F} .

To close this section, we are going to introduce a sort of “generalized Dulac transform” (or maybe “compounded Dulac transform”) for the singularities of the foliation \mathcal{F} . This material will be needed in Section 6 since the singularities of the initial foliation \mathcal{F} (as in the statement of Theorem A in the Introduction) may be degenerate. Also it should be pointed out that Proposition (4.1) is not used in the following discussion although it will be necessary in Section 6. In fact, the role played by Proposition (4.1) in Section 6 amounts to guaranteeing that the situation considered in the discussion below always occurs. In particular, this will enable us to consider the “generalized Dulac transform”, cf. below.

To explain our concern with this “generalized Dulac transform”, consider the local situation given by a singularity of \mathcal{F} that belongs to the Siegel domain. Let λ_1, λ_2 be the eigenvalues of \mathcal{F} at p and suppose that $\lambda_1 > \lambda_2$. Suppose in addition that p lies away from the divisor $(\omega)_0 \cup (\omega)_\infty$ of zeros and poles of ω , where ω stands for a meromorphic 1-form defining \mathcal{F} . Let S_1, S_2 denote the separatrices of \mathcal{F} at p that are respectively tangent to the eigendirections associated to λ_1, λ_2 . According to the discussion in Section 3, the restriction of \mathcal{H} to S_1 consists of trajectories converging to p . Similarly, the restriction of \mathcal{H} to S_2 consists of trajectories emanated from p . Thus, if l is a segment of \mathcal{H} -trajectory passing near p , the Dulac transform defined by means of l behaves as a contraction (cf. Section 3 and Lemma 3.6). The existence of this contraction is therefore consistent with the principle of producing “contractive holonomy” by following the trajectories of \mathcal{H} . However, if S_1, S_2 are contained in the divisor $(\omega)_0 \cup (\omega)_\infty$, then the orientation of \mathcal{H} around p may be “unnatural” in the sense that the Dulac transform induced by a segment of \mathcal{H} -trajectory as above actually behaves as an expansion (cf. Lemma 3.6). The tension between contraction along the leaves of \mathcal{H} and expansion for certain Dulac transforms would prevent us from guaranteeing the existence of a contractive holonomy map in a suitable sense. It is to remedy this situation that “generalized Dulac transforms” will be introduced. The aim of their study is show that contraction eventually prevails.

Without loss of generality, we can assume that \mathcal{F} is a foliation with reduced singularities defined on a certain compact surface. We also fix a non-closed meromorphic 1-form ω defining \mathcal{F} (which is supposed to exist in our case). Let $(\omega)_0^{\perp\mathcal{F}}$ (resp. $(\omega)_\infty^{\perp\mathcal{F}}$) be the subdivisor of $(\omega)_0$ (resp. $(\omega)_\infty$) consisting of those irreducible components of $(\omega)_0$ (resp. $(\omega)_\infty$) that are *not* invariant by \mathcal{F} . As before we set $(\omega)_0^\mathcal{F} = (\omega)_0 \setminus (\omega)_0^{\perp\mathcal{F}}$ and $(\omega)_\infty^\mathcal{F} = (\omega)_\infty \setminus (\omega)_\infty^{\perp\mathcal{F}}$. Let E be a connected component of $(\omega)_0^\mathcal{F} \cup (\omega)_\infty^\mathcal{F}$. Modulo performing finitely many blow-ups, we can assume without loss of generality that $(\omega)_0^{\perp\mathcal{F}}$ (resp. $(\omega)_\infty^{\perp\mathcal{F}}$) intersects E only at regular points of \mathcal{F} (cf. Lemma 5.1 in Section 5 for a detailed explanation of this procedure). The irreducible components of E are going to be denoted by D_1, \dots, D_n .

Let us now consider a leaf L of \mathcal{F} that accumulates on a singularity $P_0 \in D_1 \subseteq E$. We suppose that P_0 belongs to the Siegel domain and that L is not locally contained in the separatrices of \mathcal{F} at P_0 . One of these separatrices, S^{P_0} , of \mathcal{F} at P_0 is transverse to E (and thus not contained in E). The other separatrix of \mathcal{F} at P_0 is obviously contained in $D_1 \subset E$. Next suppose we are given a sequence of singularities of \mathcal{F} in E verifying the two conditions below:

1. Each singularity belongs to the Siegel domain.
2. Each singularity corresponds to the intersection of two irreducible components of E (recall that E is already totally invariant by \mathcal{F}).

The above mentioned sequence of singularities will be denoted by $\{p_1, \dots, p_k\}$. We suppose that p_k belongs to a component D_l of E (note that l may differ from k since the Dynkin diagram of E is allowed to contain loops). Finally one still has a singularity $P_1 \in D_l$ belonging to the Siegel domain and having a separatrix S^{P_1} transverse to E (the other separatrix of \mathcal{F} at P_1 being contained in $D_l \subset E$). Let Σ_0, Σ_1 be local transverse sections at points $z_0 \in S^{P_0}$ and $z_1 \in S^{P_1}$, respectively. Denote by μ_0, μ_1 measures on Σ_0, Σ_1 representing T over these transversals (as in Lemma 3.7).

We want to define the “generalized Dulac transform” GDul from a domain $W \subset \Sigma_0$ to Σ_1 . This can naturally be done by composing the (ordinary) Dulac transforms associated to the singularities $P_0, p_1, \dots, p_k, P_1$. Proposition (4.6) below makes this definition precise and collect the properties of GDul that are going to be used in Section 6.

Keeping the preceding notations, we have two further assumptions.

3. All the singularities $P_0, p_1, \dots, p_k, P_1$ satisfy the condition $\lambda_1(1+a) - \lambda_2(1+b) \neq 0$ of Lemma (3.1) and subsequent ones in Section 3.
4. The trajectory l_{z_0} of \mathcal{H} through $z_0 = \Sigma_0 \cap S^{P_0}$ converges to P_0 . It then continues to p_1 and from p_1 to p_2 and so on until it reaches P_1 . From P_1 this trajectory leaves E (and thus a small tubular neighborhood of E) by following the separatrix S^{P_1} . This trajectory is also assumed to pass through $z_1 = \Sigma_1 \cap S^{P_1}$.

For a detailed definition of the trajectories of \mathcal{H} “passing through singularities in the Siegel domain”, the reader is referred to the discussion carried out in Section 5. The definition of GDul simply consists of the composition of Dulac transforms associated to the singularities in question with ordinary holonomy maps associated to the segments of the leaf of \mathcal{H} between two such singularities. Finally we have:

Proposition 4.6 *Under the preceding assumption, there is $1 > \lambda > 0$ with the following properties:*

1. If $V_0 \subset \Sigma_0$ is a sector of angle less than $2\pi\lambda$ and sufficiently small radius, then $\text{GDul} : V_0 \rightarrow \Sigma_1$ is well-defined and one-to-one.
2. For $v \in V_0$, one has $\|\text{GDul}(v)\| \sim O(\|v\|^{1/\lambda})$. Therefore GDul is a contraction for $\|v\|$ small.
3. One has $\mu_0(V_0) = \mu_1(\text{GDul}(V_0))$, provided that the radius of V_0 is small enough.

Proof: The statement is clear if the divisor E is empty as an already mentioned consequence of the combination of Lemmas (3.1), (3.6) and (3.7).

Consider now the case $k = 0$, i.e. both P_0 and P_1 belong to D_1 . Denote by λ_1^0, λ_2^0 (resp. λ_1^1, λ_2^1) the eigenvalues of \mathcal{F} at P_0 (resp. P_1) where λ_2^0 (resp. λ_2^1) is the eigenvalue associated to the eigendirection defined by D_1 . Now let $b \in \mathbb{Z}$ be the order of D_1 as a component of the divisor of zeros and poles of ω . Note that the separatrices S^{P_0}, S^{P_1} are not locally contained in the support of this divisor since they are transverse to E (cf. the definition of E). Hence there are local coordinates (u, v) (resp. (w, v)) around P_0 (resp. S^{P_1}) in which ω can be written as

$$\omega = v^b [\lambda_2^0 u(1 + \text{h.o.t.})dv + \lambda_1^0 v(1 + \text{h.o.t.})du], \quad (18)$$

$$\omega = v^b [\lambda_2^1 w(1 + \text{h.o.t.})dv + \lambda_1^1 v(1 + \text{h.o.t.})dw] \quad (19)$$

where $\{v = 0\} \subset D_1$. Since the trajectories of \mathcal{H} converge towards P_0 , Lemma (3.1) ensures that $\lambda_2^0 - \lambda_1^0(b+1) < 0$. Similarly, because those trajectories also leave P_1 along S^{P_1} , Lemma (3.1) provides, in addition, that $\lambda_2^1 - \lambda_1^1(b+1) > 0$. On the other hand, recall that eigenvalues are defined only up to a multiplicative constant, so that we can set $\lambda_2^0 = \lambda_1^1$. It then results that $\lambda_1^0 > \lambda_1^1$. The corresponding generalized Dulac transform, however, clearly satisfies $\|\text{GDul}(1, u)\| = O\|u\|^{\lambda_1^0/\lambda_1^1}$. Therefore GDul has the desired contracting behavior for $\|u\|$ small. The second part of the statement can directly be checked. Indeed, there are only two cases according to whether or not $\lambda_2^0 \geq \lambda_1^0$ (in any case we have $\lambda_2^1 \geq \lambda_1^1$ provided that $b \geq 0$). This verification is left to the reader.

Let us now consider the case $k = 1$. The new element appearing in this situation is the singularity $p_1 = D_1 \cap D_2$. Keeping similar notations, let b_1 (resp. b_2) denote the order of D_1 (resp. D_2) as a component of the divisor of zeros and poles of ω . The eigenvalues of \mathcal{F} at P_1, P_2 are still denoted as before. Finally let Λ_1 (resp. Λ_2) be the eigenvalue of \mathcal{F} at p_1 associated to the eigendirection given by D_1 (resp. D_2). Around P_0 , there are local coordinates (u, v) where ω is given as in (18) (with $b = b_1$). Around P_1 , we have local coordinates (w, t) , $\{t = 0\} \subset D_2$, where $\tilde{\omega}$ becomes

$$\omega = t^{b_2} [\lambda_2^1 w(1 + \text{h.o.t.})dt + \lambda_1^1 v(1 + \text{h.o.t.})dw]. \quad (20)$$

Once again Lemma (3.1) gives us that $\lambda_2^0 - \lambda_1^0(b_1 + 1) < 0$ and $\lambda_2^1 - \lambda_1^1(b_2 + 1) > 0$. Finally, in the coordinates (v, t) around p_1 , we obtain

$$\omega = b^{b_1} t^{b_2} [\Lambda_1 t(1 + \text{h.o.t.})dv + \Lambda_2 v(1 + \text{h.o.t.})dt].$$

Thanks to Lemma (3.1), we know that $\Lambda_1(b_2 + 1) - \Lambda_2(b_1 + 1) > 0$. To conclude, we first observe that we can set $\Lambda_1 = \lambda_2^0$ and $\Lambda_2 = \lambda_2^1$ since these eigenvalues are defined only up to a multiplicative constant. Therefore one has

$$\lambda_1^0(b_1 + 1)(b_2 + 1) > \lambda_2^0(b_2 + 1) > \lambda_2^1(b_1 + 1) > \lambda_1^1(b_1 + 1)(b_2 + 1)$$

so that $\lambda_1^0 > \lambda_1^1$. In other words, the generalized Dulac transform has the contracting behavior indicated in the statement. Again the verification of item 3 is left to the reader.

The general case of $k \in \mathbb{N}$ now follows easily by induction. \square

Actually our proof yields a slightly more general result. To state it let us drop Condition 3 above, i.e. the singularities $P_0, p_1, \dots, p_k, P_1$ need no longer to satisfy the condition $\lambda_1(1+a) - \lambda_2(1+b) \neq 0$. If p_i is a (Siegel) singularity at which we have $\lambda_1(1+a) - \lambda_2(1+b) = 0$, then the restrictions of ω_1 to the local separatrices of \mathcal{F} at p_i are holomorphic on a neighborhood of p_i . This setting includes the case in which the restriction of ω_1 to one (or to both) of these separatrices vanishes identically. Our purpose here is to allow the Dulac transform corresponding to $p_i = D_i \cap D_{i+1}$ to be considered (with orientation going from D_i to D_{i+1}) as a component in the constitution of the generalized Dulac transform. The reader will note that the occasional use of the Dulac map in question is consistent with the contents of Remark 3.8 and it will further be detailed in the next section. Naturally away from the singularities that fail to fulfill the condition $\lambda_1(1+a) - \lambda_2(1+b) \neq 0$ we shall always follow the trajectories of \mathcal{H} . Then the proof of Proposition 4.6 can be repeated word-by-word to provide:

Corollary 4.7 *Under the preceding assumption the statement of Proposition 4.6 still holds except that now $1 \geq \lambda > 0$. Besides if $\lambda = 1$ then GDul is defined on every sector $V_0 \subset \Sigma_0$ of angle less than 2π (and sufficiently small radius). In the latter case the generalized Dulac transform GDul is asymptotically flat at the “origin of Σ_0 ”.*

5 Topological dynamics of the trajectories of \mathcal{H}

In the preceding two sections, we have studied the local behavior of \mathcal{H} around singularities of \mathcal{F} . It is now time to make global considerations on these trajectories.

In what follows we consider a holomorphic foliation \mathcal{F} given by a globally defined meromorphic form ω on a compact surface M . As always we suppose that ω is not closed and that \mathcal{F} admits an invariant *diffuse* positive closed current T . Again $\text{supp}(T)$ will denote the support of T . Thanks to Seidenberg Theorem, we can assume without loss of generality that the singularities of \mathcal{F} are all reduced. By virtue of Proposition (4.1) this, in fact, implies that the singularities of \mathcal{F} in $\text{supp}(T)$ either belong to the Siegel domain or are irrational foci. It is also known that a singularity of \mathcal{F} in $\text{supp}(T)$ belonging to the Siegel domain is automatically linearizable provided that the quotient of its eigenvalues is rational.

As already explained, our strategy consists of following the trajectories of \mathcal{H} with the purpose of guaranteeing a “contractive behavior for the corresponding holonomy maps”. If “enough contraction” is obtained then we should be able to conclude that T is the current of integration over a compact leaf (cf. for example Lemma 2.4). It should be noted however that there are many paths, other than trajectories of \mathcal{H} , that tend to produce contraction for the corresponding holonomy maps of \mathcal{F} . These include, for example, the trajectories of \mathcal{H}^θ , $-\pi/2 < \theta < \pi/2$, or suitable combinations of those. Therefore there is a large amount of flexibility to choose “deformed trajectories” when a trajectory of \mathcal{H} approaches a singularity such as a saddle point.

Before giving precise definitions of what is meant by “deformed trajectory” or by “trajectory of finite length”, we shall perform a few reductions in our setting so as to make the subsequent discussion more transparent. Let then ω, \mathcal{F} be as above. Denote by $(\omega)_0^\mathcal{F}$ the sub-divisor consisting of those irreducible components of $(\omega)_0$ that are invariant under \mathcal{F} . Similarly set $(\omega)_0^{\perp\mathcal{F}} = (\omega)_0 \setminus (\omega)_0^\mathcal{F}$. Denoting by $(\omega)_\infty$ the divisor of poles of ω , the subdivisors $(\omega)_\infty^\mathcal{F}$ and $(\omega)_\infty^{\perp\mathcal{F}}$ are analogously defined and so are the divisors $(\omega_1)_0^\mathcal{F}, (\omega_1)_0^{\perp\mathcal{F}}$. Let us remind the reader

that $(\omega_1)_\infty$ is contained in $(\omega)_0^{\perp \mathcal{F}} \cup (\omega)_\infty^{\perp \mathcal{F}}$ so that it has no component invariant by \mathcal{F} , cf. Lemma 2.5.

The next lemma allows us to assume some standard “normalization” conditions.

Lemma 5.1 *Modulo performing finitely many blow-ups, the conditions below are always satisfied:*

1. *The singular set $\text{Sing}(\mathcal{F})$ of \mathcal{F} is disjoint from $(\omega)_0^{\perp \mathcal{F}} \cup (\omega)_\infty^{\perp \mathcal{F}}$ as well as from $(\omega_1)_0^{\perp \mathcal{F}}$.*
2. *Every irreducible component of $(\omega)_0$, $(\omega)_\infty$ and of $(\omega_1)_0$ is smooth.*
3. *The divisor of zeros $(\omega)_0$ does not intersect the divisor of poles $(\omega)_\infty$ at regular points of \mathcal{F} .*
4. *Two distinct irreducible components of $(\omega)_0^{\perp \mathcal{F}}$ (resp. $(\omega)_\infty^{\perp \mathcal{F}}$, $(\omega_1)_0^{\perp \mathcal{F}}$) are disjoint.*
5. *\mathcal{F} is transverse to every irreducible component of $(\omega)_0^{\perp \mathcal{F}} \cup (\omega)_\infty^{\perp \mathcal{F}}$ or of $(\omega_1)_0^{\perp \mathcal{F}}$.*

Proof. It is clear that the singularities of \mathcal{F} can be supposed to be reduced (Seidenberg’s theorem). Similarly the irreducible components of $(\omega)_0$, $(\omega)_\infty$ and of $(\omega_1)_0$ can easily be made smooth.

To show that $\text{Sing}(\mathcal{F})$ can be made disjoint from $(\omega)_0^{\perp \mathcal{F}} \cup (\omega)_\infty^{\perp \mathcal{F}}$, let \mathcal{C} be a local branch of an irreducible component of $(\omega)_0^{\perp \mathcal{F}} \cup (\omega)_\infty^{\perp \mathcal{F}}$ passing through $p \in \text{Sing}(\mathcal{F})$. By assumption \mathcal{C} is not invariant by \mathcal{F} so that it has a contact of finite order with the actual separatrices of \mathcal{F} at p . By blowing-up \mathcal{F} at p , the new singularities appearing in the exceptional divisor $\pi^{-1}(p)$ have their positions determined by the tangent spaces at p to the local separatrices of \mathcal{F} . Therefore, after finitely many repetitions of this procedure, the proper transform of \mathcal{C} will no longer pass through any of the resulting singularities of the blown-up foliation. A similar argument applies to the divisor $(\omega_1)_0^{\perp \mathcal{F}}$. Note also that, in the course of performing the mentioned blow-ups, the “new components” of $(\omega)_0$, $(\omega)_\infty$ and of $(\omega_1)_0$ that may have been introduced are all contained in the exceptional divisor. Hence they are invariant by the corresponding foliation, i.e. they are not contained in $(\omega)_0^{\perp \mathcal{F}} \cup (\omega)_\infty^{\perp \mathcal{F}} \cup (\omega_1)_0^{\perp \mathcal{F}}$.

The remaining “reductions” are based on the following remark: if a regular point of a foliation is blown-up, then the new foliation still leaves the exceptional divisor invariant. Furthermore this exceptional divisor contains a unique singularity of the blown-up foliation. This singularity is conjugate to the linear singularity with eigenvalues 1, -1 .

Consider now a point $p \in M$ regular for \mathcal{F} where $(\omega)_\infty$ intersects $(\omega)_0$ and let \mathcal{F} be blown-up at p . As before, after finitely many blow-ups, the proper transforms of $(\omega)_\infty$ and $(\omega)_0$ will be separated. The components added by these blow-ups are all contained in the exceptional divisor and thus are invariant by the corresponding foliation. In particular, if we just wanted to ensure that $(\omega)_\infty^{\perp \mathcal{F}}$ does not intersect $(\omega)_0^{\perp \mathcal{F}}$ at a regular point this would be enough. For the general case, it suffices to note that the order of the exceptional divisor resulting from a single blow-up is the difference between the orders of the components of $(\omega)_\infty$ and of $(\omega)_0$ that pass through the center of the blow-up. Thus after finitely many repetitions, there will appear a exceptional divisor which is *regular for ω* in the sense that it is not contained in either $(\omega)_\infty$ or $(\omega)_0$. This leads to the verification of item 3. The same reasoning allows us to obtain item 4 as well.

Finally, as to item 5, let D be an irreducible component of $(\omega)_0^{\perp \mathcal{F}} \cup (\omega)_\infty^{\perp \mathcal{F}}$ or of $(\omega_1)_0^{\perp \mathcal{F}}$. We need to check that \mathcal{F} can be made transverse to D . Thanks to the preceding items, we can assume that $D \cap \text{Sing}(\mathcal{F}) = \emptyset$. Next observe that the number of tangencies between \mathcal{F} and D

is finite since D is not invariant by \mathcal{F} (and of course every tangency has finite contact). Thus once again we only need to blow-up tangency points sufficiently many times. As always the exceptional divisors added in the procedure are all invariant by the foliation and thus do not destroy the previous “reductions”. This completes the proof of the lemma. \square

The behavior of \mathcal{H} near points in $(\omega_1)_\infty = (\omega)_0^{\perp\mathcal{F}} \cup (\omega)_\infty^{\perp\mathcal{F}}$ is clear. However the behavior of \mathcal{H} near points in $(\omega_1)_0$ needs further comments and, in particular, leads to the notion of “deformed trajectory”. It is convenient to identify three *critical regions* where the trajectories of \mathcal{H} will be allowed to be deformed. These are as follows.

First critical region: The divisor $(\omega_1)_0^{\perp\mathcal{F}}$.

Let $C_j^0, j = 1, \dots, l$, denote the irreducible components of the zero divisor $(\omega_1)_0^{\perp\mathcal{F}}$ of ω_1 . Recalling that every C_j^0 is smooth and transverse to \mathcal{F} , we can find a small “tubular neighborhood” \mathcal{V}_j of C_j^0 whose boundary $\partial\mathcal{V}_j$ is still transverse to \mathcal{F} , for every $j = 1, \dots, l$. Besides, for j fixed, we also assume that the intersection of \mathcal{V}_j with $(\omega_1)_0$ is reduced to C_j^0 (cf. item 4 of Lemma 5.1). Finally set $\mathcal{V} = \bigcup_{j=1}^l \mathcal{V}_j$. If $p \in \partial\mathcal{V}$, we can suppose without loss of generality that the leaf L_p of \mathcal{F} through p “locally slices” \mathcal{V} into a connected disc. The collection of these “discs” form the fibers of a differentiable submersion $\mathcal{V} \rightarrow (\omega_1)_0$.

Next let $p \in \partial\mathcal{V}$ and let $D_p \subset L_p$ be the above mentioned disc, i.e. D_p is the connected component containing p of $L_p \cap \mathcal{V}$. The structure of the trajectories of \mathcal{H} on D_p is described by Lemma 2.8. Denoting by $\mathcal{H}|_{D_p}$ the restriction of \mathcal{H} to D_p , it follows the existence of $2m$ separatrices for $\mathcal{H}|_{D_p}$ at $Q \simeq D_p \cap (\omega_1)_0, m \geq 2$. These separatrices are divided into two groups. Namely there are m separatrices over which one converges to Q by moving in the sense of their orientation (these separatrices are said *to approach* Q). The remaining m separatrices are such that one converges to Q by moving in the sense opposite to their orientation (these separatrices are said *to leave* Q). In addition, the total picture is symmetric by a rotation group of order m .

In this situation, if l is for example a separatrix approaching Q , we allow l to be continued in the “future” by following one of the separatrices of Q that leaves Q . The chosen separatrix can for example be one of the two separatrices that are “closest” to l but this is not necessary. More generally every trajectory of \mathcal{H} entering the neighborhood \mathcal{V} is allowed to be continued by “following” any of the separatrices of \mathcal{H} leaving Q .

Second critical region: Siegel singularities of \mathcal{F} such that the restriction of ω_1 to its local separatrices is holomorphic.

Fixed a Siegel singularity as above, let V_0 be a neighborhood of it similar to the one defined before the statement of Lemma 3.6. In particular we have sets $A = \{(u, v) \in \mathbb{C}^2 ; |u| = \epsilon_1 \text{ and } |v| < \epsilon_2\}$ and $B = \{(u, v) \in \mathbb{C}^2 ; |v| = \epsilon'_2 \text{ and } |u| < \epsilon'_1\}$. To fix notations let l be an oriented trajectory of \mathcal{H} entering this neighborhood at an intersection point of l and A (i.e. l is “close” to the axis $\{v = 0\}$). Denote by L the leaf of \mathcal{F} containing l . Since the restriction of ω_1 to $\{v = 0\}$ is holomorphic at the origin, the trajectory l can be deformed *inside* L to avoid crossing the set A (i.e. this trajectory can be deformed so as to stay away from the singularity itself). This deformation is similar to the deformations performed in the case of saddle singularities of \mathcal{H} that appear in connection with the divisor $(\omega_1)_0^{\perp\mathcal{F}}$. In particular the continuation of l will stay “close to $\{v = 0\}$ ” during the procedure. In fact, this trajectory will leave the singularity by “following” one of the separatrices of $\mathcal{H}|_{\{v=0\}}$ that leave the singularity in question (where $\mathcal{H}|_{\{v=0\}}$ stands for the restriction of \mathcal{H} to $\{v = 0\}$). Another possibility to defined continuations for l is to let l enter the neighborhood of the mentioned singularity and then use the corresponding Dulac transform to continue l as a trajectory of \mathcal{H} that is now “close to $\{u = 0\}$ ”. In this case

the desired continuation of l will be “close” to one of the separatrices of $\mathcal{H}|_{\{u=0\}}$ oriented so as to leave the mentioned singularity (where $\mathcal{H}|_{\{u=0\}}$ stands for the restriction of \mathcal{H} to $\{u = 0\}$). Summarizing it can be said that a trajectory l of \mathcal{H} intersecting the set A admits all the above mentioned continuations.

Remark 5.2 In the preceding two types of critical regions the “deformation” of the trajectory l consists of adding to it a “small” segment of trajectory of \mathcal{H}^\perp . By construction these pieces of \mathcal{H}^\perp -trajectories have length bounded by a “small constant” and besides they are strictly comprised between two “genuine” segments of \mathcal{H} -trajectories whose lengths are bounded from below by positive constants depending solely on \mathcal{F} , M . As a consequence these “deformations” do not disrupt the global contractive nature of holonomy maps of \mathcal{F} defined by means of “deformed trajectories of \mathcal{H} ”. We shall return to this point below.

Remark 5.3 Besides singularities belonging to the Siegel domain also irrational focus singularities may be considered. Recall that an irrational focus singularity is linearizable and hence it possesses exactly two separatrices. These separatrices are smooth and may be chosen as the coordinate axes in the linearizing coordinates. The fact that the quotient between the eigenvalues of these singularities cannot be a rational number implies that the only way in which the restriction of ω_1 to these separatrices may be holomorphic occurs when both separatrices are components with multiplicity 1 of $(\omega)_\infty$. This case will rarely occurs, but if it does, the trajectories of \mathcal{H} will be deformed so as to avoid the singularity in the same way it may be done for analogous Siegel singularities. Since irrational foci have no associated Dulac transforms only this type of continuation will be allowed in the present case.

Third critical region: The divisor $(\omega_1)_0^\mathcal{F}$.

By construction the support of the divisor $(\omega_1)_0^\mathcal{F}$ consists of (irreducible) curves invariant by \mathcal{F} . Let C denote one of these curves. Then the restriction of ω_1 to C vanishes identically so that it does not define any real foliation on C . Nonetheless Poincaré Lemma can still be applied to this situation. In fact, let $c : [0, 1] \rightarrow C$ be a path contained in C and consider local transverse sections $\Sigma_{c(0)}$, $\Sigma_{c(1)}$ through $c(0)$, $c(1)$ respectively. If the sections $\Sigma_{c(0)}$, $\Sigma_{c(1)}$ are parameterized as indicated in Section 2.3, then the holonomy map $\text{Hol}(c) : \Sigma_{c(0)} \rightarrow \Sigma_{c(1)}$ obtained from c and \mathcal{F} is such that $[\text{Hol}(c)]'(0) = 1$. In particular the usual holonomy group associated to the “leaf” C with respect to \mathcal{F} is entirely constituted by local diffeomorphisms tangent to the identity.

Since the foliation \mathcal{H} is not defined on C , we shall allow every (“minimizing geodesic”) path joining two points of C with length less than the diameter of C to be used to continue a given trajectory l of \mathcal{H} . Here both “length” of the path and “diameter” of C arise from fixing once and for all some auxiliary Hermitian metric on M .

To better explain the above definition, consider two Siegel singularities P, Q of \mathcal{F} lying in C . Denote by S_p (resp. S_q) the local separatrix of \mathcal{F} transverse to C at P (resp. Q). Also fix neighborhoods V_p, V_q of P, Q as in the case of discussed in the second critical region. Since ω_1 vanishes identically on C , it follows that the restriction of ω_1 to S_p (resp. S_q) is holomorphic. If $l \subset S_p$ is a trajectory of \mathcal{H} that enters V_p , then l can be continued as a trajectory l' of \mathcal{H} contained in S_q and oriented so as to leave the neighborhood V_q . A similar convention applies to trajectories l of \mathcal{H} that are not contained in S_p but that still enters the neighborhood V_p of P . A continuation l' for the trajectory l will be such that l, l' are contained in the same global leaf L of \mathcal{F} and l' leaves the neighborhood V_q of Q . In other words the continuation of these trajectories

can be pictured as if the curve C were collapsed into a single point (heuristically imagined as a Siegel singularity whose separatrices would be S_p, S_q). Then the mentioned continuation would be defined as in the case of the second critical region discussed above.

Remark 5.4 In line with Remark 5.3, the use of Dulac transforms to follow a component C of $(\omega_1)_0^{\mathcal{F}}$ as above is only possible at a Siegel singularity i.e. no irrational focus singularity lying in C will be associated with continuation of trajectories by means of Dulac transforms.

We are now ready to define *global deformed trajectories of \mathcal{H}* . Away from a fixed neighborhood of the three critical regions previously discussed, a deformed trajectory must agree with an ordinary trajectory of \mathcal{H} . However if a trajectory l of \mathcal{H} enters a critical region, then it possesses all the corresponding continuations mentioned above. As a consequence, every possible continuation of l will eventually leave the critical region in question and become again an ordinary trajectory of \mathcal{H} . In particular, given $p \in M$, the deformed trajectory of \mathcal{H} through p is in general not uniquely determined. It is convenient to think of it not as a “single path” but as a collection of paths that ramifies whenever one of its branches enters a critical region. In other words, the \mathcal{H} -trajectory through p is in general not a single path but rather a collection of paths (or branches) that are allowed to ramify at the critical regions.

Similar definitions apply if we decide to follow a (deformed) trajectory l of \mathcal{H} in the direction opposite to its orientation (i.e. when the “past” of l is considered). More generally for $\theta \in (-\pi/2, \pi/2)$ fixed, the deformed trajectories of \mathcal{H}^θ are analogously defined and the same remark concerning orientation can be done to define their “continuations in the past”.

To fully define what will be understood by a deformed trajectory of \mathcal{H} (or \mathcal{H}^θ we still need to clarify what to do when an ordinary trajectory becomes close to the remaining “singularities” of \mathcal{H} (or of \mathcal{H}^θ). However before doing this, it is important to point out that deformed trajectories as considered above are such that the corresponding holonomy maps of \mathcal{F} still keep the contractive behavior characteristic of ordinary trajectories of \mathcal{H} (or \mathcal{H}^θ for $\theta \in (-\pi/2, \pi/2)$). As in Section 2.3, recall that we have fixed an auxiliary Hermitian metric on M so that it is possible to consider the length of paths contained in M . For those paths whose images are contained in leaves of \mathcal{F} , their resulting lengths are also comparable with the sum of the lengths of their representatives in a fixed foliated atlas of M , cf. Section 2.3. Next let $\theta \in (-\pi/2, \pi/2)$ be fixed and consider a path $c : [0, 1] \rightarrow L \subset M$ parameterizing a segment of deformed trajectory of \mathcal{H} (resp. \mathcal{H}^θ for fixed $\theta \in (-\pi/2, \pi/2)$) as above. Thus c can be viewed as a concatenation of paths c^i that either are contained in a critical region or are segments of (ordinary) \mathcal{H} -trajectories (resp. \mathcal{H}^θ -trajectories) away from the critical regions and from the remaining singularities of $\mathcal{H}, \mathcal{H}^\theta$. In the latter case, the length of c^i is bounded from below by a positive constant. On the other hand a path c^i whose image is contained in a critical region is such that c^{i-1} and c^{i+1} parameterize an ordinary segment of $\mathcal{H}, \mathcal{H}^\theta$ (lying in a compact part of the complement of the critical regions and of the remaining singularities of $\mathcal{H}, \mathcal{H}^\theta$). Furthermore these paths c^i are such that their length is uniformly bounded and, besides, the holonomy map of \mathcal{F} obtained by means of c^i (with respect to suitable transverse sections parameterized as indicated in Section 2.3) is a holomorphic diffeomorphism whose linear part has modulo equal to 1. Then the preceding discussion can be summarized by Proposition 5.5 below, which ensures the exponential decay of the norm of the derivative at $c(0)$ with the length of c .

Proposition 5.5 *Consider a path $c : [0, 1] \rightarrow L$ that parametrizes a segment of deformed trajectory of \mathcal{H} or, more generally, of \mathcal{H}^θ ($-\pi/2 < \theta < \pi/2$). Then there are constants C, k depending*

solely on θ (for M, \mathcal{F}, ω and the auxiliary Hermitian metric fixed) such that the estimate below holds

$$|(\text{Hol}(c))'(0)| \leq C \exp(-k \text{length}(c)/2), \quad (21)$$

where $\text{Hol}(c)$ stands for the holonomy map of \mathcal{F} induced by c . \square

Let us now complete the definition of deformed trajectories for \mathcal{H} . First note that the “remaining singularities” of $\mathcal{H}, \mathcal{H}^\theta$ are provided by either singular points of \mathcal{F} (different from Siegel singularities since these were already taken into account) and by the divisors $(\omega)_0^{\perp \mathcal{F}}$ and $(\omega)_\infty^{\perp \mathcal{F}}$. For the purposes of this paper however, Proposition 4.1 allows us to rule out hyperbolic singularities as well as saddle-node singularities from the discussion below.

Unless otherwise stated, in what follows we shall simply say *trajectory of \mathcal{H} (resp. \mathcal{H}^θ)* instead of “deformed trajectory of \mathcal{H} (resp. \mathcal{H}^θ)”. Hence for $p \in \mathcal{K}$, let l_p denote the trajectory of \mathcal{H} through p (in precise words, this means a deformed trajectory of \mathcal{H} through p) and consider the leaf L of \mathcal{F} containing l_p . Let us first introduce the notion of *endpoint* for l_p . The trajectory l_p is said to have an *endpoint* at a point $q \in M$ if one of the following possibilities hold:

- $q \in (\omega)_0^{\perp \mathcal{F}}$ is a sink of $\mathcal{H}|_L$ and $\bar{l}_p^+ = q$.
- $q \in (\omega)_\infty^{\perp \mathcal{F}}$ is a source of $\mathcal{H}|_L$ and $\bar{l}_p^- = q$.
- q is a sink-irrational focus (resp. source-irrational focus) singularity of \mathcal{F} to which l_p converges (resp. from which l_p is emanated, cf. Lemma 4.4 and Remark 4.5).

Similar definitions apply to the case of \mathcal{H}^θ -trajectories, $\theta \in (-\pi/2, \pi/2)$. Sometimes we shall use the expressions *future end* (resp. *past end*) to refer to the cases above which are concerned with a sink-like (resp. source-like) endpoint of l_p . To define the trajectory of l_p of \mathcal{H} through p we start with the ordinary trajectory of \mathcal{H} through p . Whenever this trajectory enters one of the above described critical regions, all its resulting ramifications are considered together as its continuations. Thus it is possibly more convenient to speak about *branches* of l_p . In this case a branch of l_p has a future endpoint if it converge to a point q of M that behaves locally as a sink for \mathcal{H} . In view of the preceding q either belongs to $(\omega)_0^{\perp \mathcal{F}}$ or it is a sink-irrational focus singularity of \mathcal{F} . Naturally we can also follow the trajectory l_p in the sense opposite to its standard orientation. In this case, we shall denote the resulting semi-trajectory by l_p^- (in certain cases where the context might be unclear, the semi-trajectories through p with the usual orientation will also be denoted by l_p^+). Past endpoints for a branch of l_p^- is then analogously defined and so are future and past endpoints for branches of the trajectories $l_p^\theta = l_p^{\theta,+}$ and $l_p^{\theta,-}$ of \mathcal{H}^θ through p .

The length of a branch of the semi-trajectory l_p^+ is defined in natural differential geometric terms for the auxiliary Hermitian metric fixed from the beginning provided that the branch is finite. Otherwise the branch is said to be of infinite length. Now the semi-trajectory l_p^+ of \mathcal{H} (resp. \mathcal{H}^θ) through p is said to be *finite* if and only the supremum of the length of all its branches is finite. In this case the number of branches of l_p^+ is itself finite so that the supremum is also attained. Once again the definition of length for the semi-trajectory l_p^- can analogously be given. Finally the deformed trajectory l_p of \mathcal{H} (resp. \mathcal{H}^θ) through p will be called finite if both semi-trajectories l_p^+ and l_p^- are so. The length of l_p will then be the maximum of the lengths of all branches contained in l_p .

Remark 5.6 If a branch of a \mathcal{H} -trajectory l_p (resp. \mathcal{H}^θ -trajectory l_p^θ) consists of a loop, possibly passing through critical regions, then this branch contains neither future nor past endpoints. It then follows that its length is infinite. This means that one is allowed to go around the loop infinitely many times what explains why the length of the loop must be considered as infinite. This is very natural as definition since the holonomy of \mathcal{F} associated to one of these trajectories is clearly hyperbolic.

In fact, with our terminology, the length of l_p is finite if and only if all branches of l_p possess both future and past ends and, in addition, the supremum of the lengths of these branches is finite. It is also clear that the above definition is invariant by blow-ups/blow-downs. Therefore the length of the trajectories of \mathcal{H} (resp. \mathcal{H}^θ) can be considered whether or not the foliation \mathcal{F} has reduced singularities. More generally, this definition makes sense whether or not the normalizing conditions of Lemma 5.1 are satisfied.

With the above terminology, the contents of Proposition 5.5 can be complemented by the following simple generalization of Theorem 2.10 that is better adapted to our needs. Let K be a compact part of the complement of the singular set of \mathcal{F} and consider a path $c : [0, 1] \rightarrow K$ parameterizing a segment of deformed \mathcal{H} -trajectory (resp. \mathcal{H}^θ -trajectory, $\theta \in (-\pi/2, \pi/2)$). Finally let $\text{Hol}(c)$ denote the holonomy map of \mathcal{F} induced by c and recall that these maps are identified with local diffeomorphisms of \mathbb{C} by means of transverse sections $\Sigma_{c(0)}, \Sigma_{c(1)}$ parameterized by ω (cf. Section 2.3). Then we have:

Theorem 5.7 *With the preceding notations there is $\delta > 0$ (depending only on K) and constants $C, k > 0$ such that the following holds:*

1. $\text{Hol}(c)$ is defined on the transverse disc $B_{c(0)}(\delta)$ of radius $\delta > 0$ about $c(0)$.
2. The image $(\text{Hol}(c))(B_{c(0)}(\delta))$ of $B_{c(0)}(\delta)$ by $\text{Hol}(c)$ is contained in a transverse disc $B_{c(1)}(r)$ of radius r about $c(1)$ where

$$r \leq C \exp(-k \text{length}(c)/2).$$

More generally if c parameterizes a segment of deformed trajectory of \mathcal{H}^θ , $\theta \in (-\pi/2, \pi/2)$, then the statement still holds, only the values of the constants δ, C, k will depend further on θ .

Proof: Fix a finite covering of K by foliated coordinates of \mathcal{F} along with transverse sections parameterized by ω as indicated in Section 2.3. According to Proposition 5.5, there are constants C_1, k_1 such that the absolute value of the derivative of $\text{Hol}(c)$ at $c(0)$ satisfies the estimate

$$|(\text{Hol}(c))'(0)| \leq C_1 \exp(-k_1 \text{length}(c)/2).$$

As in [B-L-M], this estimate allows us to show that $\text{Hol}(c)$ is defined on a uniform domain $B_{c(0)}(\delta)$. To check the rest of the statement, note that $\text{Hol}(c)$ is univalent on its domain of definition. Thus modulo reducing this domain, K oebe's theorem (cf. [Po]) can be applied to ensure that $\text{Hol}(c)$ has "bounded distortion" so that the diameter of its image can be estimate from the value of its derivative at $c(0)$. This completes the proof of the theorem for $\theta = 0$. The general case is however totally analogous. \square

From now to the rest of the paper we fix a closed set $\mathcal{K} \subseteq \text{supp}(T)$ that is minimal for \mathcal{F} . Denote by $\mathcal{H}_{\mathcal{K}}$ (resp. $\mathcal{H}_{\mathcal{K}}^\theta$) the restriction of \mathcal{H} (resp. \mathcal{H}^θ) to \mathcal{K} . As it is usually the case, in the sequel the word "trajectory" actually means "deformed trajectory". Let us close this section with the following proposition:

Proposition 5.8 *The following alternative holds:*

- *There is a uniform constant C (resp. C^θ) such that the length of every trajectory of $\mathcal{H}_\mathcal{K}$ (resp. $\mathcal{H}_\mathcal{K}^\theta$) is less than C (resp. C^θ).*
- *There is a non-empty compact set $\mathcal{K}^0 \subseteq \mathcal{K}$, invariant by \mathcal{H} (resp. \mathcal{H}^θ), where all the corresponding trajectories of (the restriction of) \mathcal{H} (resp. \mathcal{H}^θ) have infinite length.*

Remark 5.9 The invariance of \mathcal{K}^0 by \mathcal{H} (resp. \mathcal{H}^θ) means that through each point of \mathcal{K}^0 there passes a branch of trajectory of \mathcal{H} (resp. \mathcal{H}^θ) which is entirely contained in \mathcal{K}^0 . Since, in general, the trajectory of \mathcal{H} (resp. \mathcal{H}^θ) through p is constituted by several branches, it may well happen that some of them are not fully contained in \mathcal{K}^0 .

Proof of Proposition (5.8): In the sequel we suppose that the conditions of Lemma 5.1 are satisfied. It suffices to check the statement for the foliation $\mathcal{H} = \mathcal{H}^0$.

Denote by $\text{Irr}_+(\mathcal{F})$ (resp. $\text{Irr}_-(\mathcal{F})$) the irrational focus singularities of \mathcal{F} in \mathcal{K} that behave as a sink (resp. source) for \mathcal{H} in the sense of Lemma (4.4). Given one such singularity p , let $B_p(\epsilon)$ be the real 3-dimensional ball of radius $\epsilon > 0$ about p . It is easy to see that \mathcal{F} is transverse to this ball for ϵ sufficiently small. In fact, also the foliation \mathcal{H} is transverse to this ball as it easily follows from Proposition (4.4). Let then $\text{Irr}_+^\epsilon(\mathcal{F})$ be the union of these balls about the points in $\text{Irr}_+(\mathcal{F})$. The set $\text{Irr}_-^\epsilon(\mathcal{F})$ is analogously defined.

Suppose that all the (deformed) \mathcal{H} -trajectories contained in \mathcal{K} are of finite length. We are going to show the existence of a uniform bound for all the corresponding lengths. If this bound did not exist, then there would be a sequence $\{l_i\}_{i \in \mathbb{N}}$ of branches of \mathcal{H} -trajectories contained in \mathcal{K} such that the sequence formed by their corresponding lengths goes off to infinity. For each i , let $c_i(a_i, b_i) \subset \mathbb{R} \rightarrow M$ be a parametrization of l_i . Naturally $c_i(a_i)$ belongs to $(\omega)_\infty \cup \text{Irr}_-(\mathcal{F})$ whereas $c_i(b_i)$ belongs to $(\omega)_0 \cup \text{Irr}_+(\mathcal{F})$. Thus we conclude that, in fact, $c_i(a_i) \in (\omega)_\infty^{\perp \mathcal{F}} \cup \text{Irr}_-(\mathcal{F})$ and $c_i(b_i) \in (\omega)_0^{\perp \mathcal{F}} \cup \text{Irr}_+(\mathcal{F})$. Modulo passing to a subsequence, we can suppose that the $c_i(a_i)$ (resp. $c_i(b_i)$) converge to a point $a \in \mathcal{K}$ (resp. $b \in \mathcal{K}$). One of the following two possibilities must occur:

1. a (resp. b) belongs to $D_\infty \subseteq (\omega)_\infty$ (resp. $b \in D_0 \subseteq (\omega)_0$) where D_∞ (resp. D_0) stands for an irreducible component of $(\omega)_\infty$ (resp. $(\omega)_0$).
2. a (resp. b) belongs to $\text{Irr}_-(\mathcal{F})$ (resp. $\text{Irr}_+(\mathcal{F})$). In this case, modulo shortening the length of c_i by a uniform small constant (cf. Proposition 4.4) we can replace a_i (resp. b_i) by a'_i (resp. b'_i) such that $c_i(a'_i) \in \text{Irr}_-^\epsilon(\mathcal{F})$ (resp. $c_i(b'_i) \in \text{Irr}_+^\epsilon(\mathcal{F})$). Therefore we can consider without loss of generality that $a \in \text{Irr}_-^\epsilon(\mathcal{F}) \cup \text{Irr}_+^\epsilon(\mathcal{F})$.

Consider the leaf L_a of \mathcal{F} through a and note that the restriction of \mathcal{H} to L_a is well-defined (it is not fully constituted by a critical region of third type). In particular, there is a trajectory l_a^+ of \mathcal{H} being emanated from a . Although this trajectory possibly consists of several branches, due to ramification at critical regions, it contains one special branch defined as follows: whenever the (branch of the) trajectory in question enters a critical region, its continuation is dictated by the continuations of the l_i 's (for i large enough). The resulting branch l_a^+ clearly has infinite length. Otherwise leaves emanated from points sufficiently close to a , and choosing appropriate ramifications at critical regions, would have bounded length and this would contradict our assumption.

The preceding discussion also shows the existence of semi-trajectories of infinite lengths provided that the first case in the statement of the proposition does not occur. Let then l^+ denote a branch of infinite length contained in some deformed trajectory in \mathcal{K} . Since \mathcal{K} is compact, the closure $\bar{l}^+ \subset \mathcal{K}$ of l^+ is not empty. Besides every semi-trajectory through a point of \bar{l}^+ is infinite, or in more accurate terms, it contains a branch of infinite length. In fact, if all branches of a deformed trajectory through a point $p \in \bar{l}^+ \subset \mathcal{K}$ were of finite length, then the above argument would imply that \bar{l}^+ intersects $(\omega)_0^{\perp \mathcal{F}} \cup \text{Irr}_+^\epsilon(\mathcal{F})$. This is however impossible since it contradicts the infinite length of l^+ . In other words, $\mathcal{K}^0 = \bar{l}^+$ satisfies the condition in the second alternative of our statement. The proposition is proved. \square

Since $\mathcal{K}^0 = \bar{l}^+$ and $(\omega)_0^{\perp \mathcal{F}} \cup \text{Irr}_+^\epsilon(\mathcal{F})$ are compact disjoint, there is a positive distance between them. Also, by construction, l^+ cannot accumulate (in the future) on $(\omega)_\infty^{\perp \mathcal{F}} \cup \text{Irr}_-^\epsilon(\mathcal{F})$ thanks to Lemma (2.7) and Lemma (4.4). Thus we obtain:

Corollary 5.10 *Suppose that the first alternative in Proposition (5.8) is not verified. Then there is a small open neighborhood V of $(\omega)_0^{\perp \mathcal{F}} \cup (\omega)_\infty^{\perp \mathcal{F}} \cup \text{Irr}_-^\epsilon(\mathcal{F}) \cup \text{Irr}_+^\epsilon(\mathcal{F})$ such that $\mathcal{K}^0 \cap V = \emptyset$. In particular all the singularities of \mathcal{F} lying in \mathcal{K}^0 are in the Siegel domain unless are irrational foci as in Remark 5.3.*

6 Invariant currents vs. infinite trajectories of \mathcal{H}

The remaining two sections are devoted to proving the theorems stated in the Introduction. We keep the context and the notations of Section 5. Recalling that \mathcal{K} stands for a minimal set of \mathcal{F} contained in the support of T , the restriction of \mathcal{H} to \mathcal{K} is going to be denoted by $\mathcal{H}_\mathcal{K}$. Let us begin by rephrasing Theorem A:

Theorem 6.1 *Let \mathcal{F} and T be as above. If \mathcal{K} does not contain a compact leaf of \mathcal{F} , then all deformed \mathcal{H} -trajectories (resp. \mathcal{H}^θ -trajectories with fixed $\theta \in (-\pi/2, \pi/2)$) in \mathcal{K} have length smaller than some positive constant Const .*

It suffices to prove the statement for $\mathcal{H} = \mathcal{H}^0$ since the generalization to \mathcal{H}^θ is very straightforward. Thus let us suppose that the lengths of the \mathcal{H} -trajectories in \mathcal{K} are not uniformly bounded. Our aim will then be to ensure the existence of an algebraic curve contained in \mathcal{K} . Since the lengths of the \mathcal{H} -trajectories in \mathcal{K} are not uniformly bounded, we can consider a compact set $\mathcal{K}^0 \subset \mathcal{K}$ satisfying the conclusions of Proposition 5.8 and Corollary 5.10. Moreover, by applying Zorn Lemma, we can assume without loss of generality that \mathcal{K}^0 is *minimal* for \mathcal{H} i.e. through every point in \mathcal{K}^0 there passes a branch of (deformed) trajectory of \mathcal{H} which is dense in \mathcal{K}^0 (the branch being obviously contained in \mathcal{K}^0). Here it is worth pointing out that the assumption that \mathcal{K}^0 is minimal is not indispensable for our discussion (and so the use of Zorn Lemma can also be avoided). In fact, it would be enough to consider an accumulation point of a \mathcal{H} -trajectory $l^+ \subset \mathcal{K}$ of infinite length that happens to be regular for \mathcal{H} . Because of Theorem 5.7 the holonomy maps of \mathcal{F} induced by (segments of) l^+ is defined on a uniform domain. Thus if l^+ accumulates on a regular point of \mathcal{H} (and of \mathcal{F}) this trajectory will be captured by the holonomy maps to which it gives rise (modulo a slight local deformation of l^+). The latter statement would be sufficient for our purposes. Yet it is simpler to assume that \mathcal{K}^0 is minimal so that a self-accumulating \mathcal{H} -trajectory l^+ can be selected.

Next let us perform on \mathcal{F} the normalizations described in Lemma 5.1. Since these transformations include the blowing-up of points, they give rise to *compact leaves* contained in the closure of the transform of \mathcal{K} . The curves obtained in this way however *do not form loops* and this will be exploited in the sequel, cf. below. More generally let $A_{\mathcal{K}}$ denote the union of all algebraic curves contained in transform of \mathcal{K} . Actually, by an abuse of notation, the closure of the proper transform of \mathcal{K} will still be denoted by \mathcal{K} . In this sense, \mathcal{K} is no longer minimal for \mathcal{F} but it satisfies the following condition: every leaf of \mathcal{F} in \mathcal{K} that is not dense in $\mathcal{K} \setminus A_{\mathcal{K}}$ is necessarily contained in $A_{\mathcal{K}}$ itself. This condition is going to be used in the sequel. To give more accurate statements, consider all irreducible compact leaves of \mathcal{F} contained in \mathcal{K} . Obviously we can assume there are finitely many D_1, \dots, D_r of those. Besides $A_{\mathcal{K}} = D_1 \cup \dots \cup D_r$. We shall say that these curves *contain a loop* if there are pairwise distinct points p_{i_1}, \dots, p_{i_s} with $p_{i_j} \in D_{i_j} \cap D_{i_{j+1}}$ for $1 \leq j < s$ and $p_{i_s} \in D_{i_s} \cap D_{i_1}$. We can now state a sharper form of Theorem (6.1).

Theorem 6.2 *Let \mathcal{F} and T be as in Theorem 6.1. Suppose that \mathcal{K} contains only finite many irreducible compact curves D_1, \dots, D_r invariant by \mathcal{F} and that these curves do not contain loops. Suppose also that the remaining leaves of \mathcal{F} are dense in $\mathcal{K} \setminus A_{\mathcal{K}}$. Then all (deformed) \mathcal{H} -trajectories in \mathcal{K} have length smaller than some positive constant Const . An analogous statement is valid for the trajectories of \mathcal{H}^θ -trajectories, $\theta \in (-\pi/2, \pi/2)$.*

Theorem 6.1 is an immediate consequence of Theorem 6.2. In fact, in the context of Theorem 6.1 (before performing the normalizations associated with Lemma 5.1), we assume that the lengths of all deformed \mathcal{H} -trajectories contained in \mathcal{K} are not uniformly bounded. In particular there is a compact set $\mathcal{K}^0 \subset \mathcal{K}$ and a self-accumulating infinite brach l^+ of a deformed \mathcal{H} -trajectory that is contained in \mathcal{K}^0 . To pass from this situation to the context of Theorem 6.2, let us now perform the normilzations of Lemma 5.1. in view of Theorem 6.2 the irreducible components of $A_{\mathcal{K}}$ must form a loop. Since the components of $A_{\mathcal{K}}$ introduced in the course of the normalization procedure in question are rational curves contained in pairwise disjoint tree-like arrangements, the only way for the components of $A_{\mathcal{K}}$ to form a loop arises from the existence of algebraic curves in the initial minimal set \mathcal{K} . Thus Theorem 6.1 follows.

The rest of this paper is ultimately devoted to the proof of Theorem 6.2 for the case $\mathcal{H} = \mathcal{H}^0$. The extension to \mathcal{H}^θ for $\theta \in (-\pi/2, \pi/2)$ will be left to the reader. For this we assume that \mathcal{F} , ω , ω_1 and so on, satisfy all the conditions in the statement of Lemma 5.1. In particular the so-called “local invariance condition” of Section 3 concerning singularities of \mathcal{F} that belong to the Siegel domain is verified. Also, modulo fixing a neighborhood \mathcal{W} of the singular set of \mathcal{F} , holonomy maps of \mathcal{F} obtained by means of (segments of) deformed \mathcal{H} -trajectories contained in the complement of \mathcal{W} must satisfy the conclusions of Theorem 5.7. Besides l^+ and \mathcal{K}^0 will always be as indicated above. Now we have:

Lemma 6.3 *We have $\mathcal{K}^0 \cap \text{Sing}(\mathcal{F}) \neq \emptyset$.*

Proof: It is a simple application of Theorem 5.7. Suppose that the statement is false. Thus modulo reducing \mathcal{W} we can assume that \mathcal{K} lies entirely in the complement of \mathcal{W} . Now consider a parametrization c for a trajectory of \mathcal{H} such that $c(0) = p \in \mathcal{K}^0$ (where p does not belong to any critical region). Fix a local transverse section Σ_p and a disc $B_p(r) \subset \Sigma_p$ as in item 1 of Theorem 5.7. Finally, for a fixed $t_0 \in \mathbb{R}_+$, let $\text{Hol}(c_{t_0})$ be the holonomy map associated to the restriction of c to $[0, t_0]$.

By construction, $\text{Hol}(c_{t_0})$ is defined on $B_p(r)$ for every t_0 . On the other hand, since the leaves of \mathcal{K} are dense in \mathcal{K}^0 , there is a sequence of times t_0^1, t_0^2, \dots going to infinity and such that $\{c(t_0^i)\}$ converges to p when $i \rightarrow \infty$. Because Σ_p is not a transverse section for \mathcal{H} at p , we cannot ensure *a priori* that $c(t_0^i)$ can be chosen in Σ_p for every $i \in \mathbb{N}$. Nonetheless, modulo performing a slight modification of the trajectories of \mathcal{H} on a neighborhood of p (similar in spirit to the “deformed” trajectories arising from the first critical regions) this assumption can be made without loss of generality.

Now, thanks to the second part of the statement of Theorem 5.7, it follows that the image of $B_p(r)$ under $\text{Hol}(c_{t_0})$ is contained in a disc of radius $r/10$ about $c(t_0^i) \in \Sigma_p$ provided that i is large enough. In other words, for i very large $\text{Hol}(c_{t_0})$ takes the disc $B_p(r)$ inside itself. This actually implies the existence of a loop with hyperbolic holonomy for \mathcal{F} . As already seen, this gives a contradiction in the present case since T is a diffuse, cf. Lemma 2.4. \square

Recall that, in principle, singularities of \mathcal{F} lying in \mathcal{K}^0 either are Siegel singularities (possibly associated to critical regions of second type) or are irrational foci as in Remark 5.3. The latter singularities are however necessarily avoided by the trajectories of \mathcal{H} as it was discussed in Section 5. Therefore irrational focus singularities can be ignored in our context and we can suppose that all singularities of \mathcal{F} lying in \mathcal{K}^0 are, in fact, Siegel singularities.

Let us now state a proposition that plays a key role in the proof of Theorem 6.2.

Proposition 6.4 *The above mentioned trajectory l^+ and set \mathcal{K}^0 can be chosen so that $l^+ \subset \mathcal{K}^0$ is closed.*

In view of the assumption about minimality of \mathcal{K}^0 with respect to \mathcal{H} , the preceding proposition actually says that \mathcal{K}^0 is reduced, in a suitable sense, to a closed trajectory l^+ . On the other hand, recall that our definition of “closed trajectory” allows l^+ to pass through singularities of \mathcal{F} lying in \mathcal{K}^0 (which are necessarily Siegel singularities as already pointed out). Indeed a closed trajectory must necessarily go through singularities of \mathcal{F} since otherwise it gives rise to a holonomy map of \mathcal{F} possessing a hyperbolic fixed point. As already seen this forces the current T to be concentrated over an algebraic curve contradicting its diffuse nature. On the other hand, the fact that a closed trajectory goes through a singularity of \mathcal{F} implies the existence of at least one saddle connexion for \mathcal{F} .

In the rest of this section Proposition 6.4 is going to be proved. In the next section we shall use it to derive the proof of Theorem 6.2 and of Theorem B in the Introduction.

As in Section 4, we consider the connected components $E = E_1, E_2, \dots$ of the compact curves invariant by \mathcal{F} and contained in \mathcal{K} . Hence each E_i consists of a number of irreducible curves $D_{i_k} \subset \{D_1, \dots, D_r\}$. Note that our terminology allows E_i to be empty i.e. reduced to a Siegel singularity that does not belong to any compact curve invariant by \mathcal{F} . To make the subsequent discussion more transparent we shall first consider the following special situation:

First Case: there is a unique connected component E .

The general case can easily be deduced from our discussion as it will be shown at the end of the section. Let us then fix a singularity $p \in E \cap \mathcal{K}^0$ (lying away from the critical regions). The next lemma allows us to suppose in addition that a segment of l^+ is contained in a local separatrix of \mathcal{F} at p .

Lemma 6.5 *There is a (deformed) semi-trajectory of $\mathcal{H}_{\mathcal{K}^0}$ contained in a separatrix of \mathcal{F} and having p as an accumulation point (where $\mathcal{H}_{\mathcal{K}^0}$ stands for the restriction of \mathcal{H} to \mathcal{K}^0). This trajectory will still be denoted by l^+ .*

Proof: Fix local coordinates (u, v) around $p \simeq (0, 0)$ in which the 1-form ω defining \mathcal{F} satisfies Equation (6). We choose u, v so that the trajectories of \mathcal{H} in $\{v = 0\}$ converge to $p \simeq (0, 0)$. Next let Σ_θ be a local transverse section passing through the point $(e^{2\pi i\theta}, 0)$, $\theta \in [0, 2\pi)$. Since $p \in \mathcal{K}^0$, Proposition (3.4) implies the existence of a sequence of points (θ_i, v_i) such that $(e^{2\pi i\theta_i}, v_i) \in \mathcal{K}^0$ and $|v_i| \rightarrow 0$. Since \mathcal{K}^0 is closed, it follows the existence of θ_∞ such that $(e^{2\pi i\theta_\infty}, 0) \in \mathcal{K}^0$. The trajectory of \mathcal{H} through this point then satisfies the required conditions. \square

Remark 6.6 Without loss of generality we can suppose that $\theta_\infty = 0$ so that $(1, 0) \in \mathcal{K}^0$. We also set $\Sigma = \Sigma_0$ and denote by l the trajectory of \mathcal{H} through $(1, 0)$ which is obviously contained in \mathcal{K}^0 .

Lemma 6.7 *l^+ is not entirely contained in E .*

Proof: Suppose for a contradiction that l^+ is entirely contained in E . Suppose that D_1 is the irreducible component of E containing p . We can assume without loss of generality that D_1 contains the whole of l^+ . Indeed, suppose that by following l^+ one passes from D_1 to another irreducible component D_2 . This passage is then made through a singularity $q_{1,2} = D_1 \cap D_2$ which belongs to the Siegel domain. By assumption the orientation of the trajectories of \mathcal{H} around $q_{1,2}$ (always given by Lemma 3.1) is such that they go from the separatrix contained in D_1 to the separatrix contained in D_2 . Hence the trajectory l^+ cannot return to D_1 through $q_{1,2}$. Because $D_1 \cap D_2 = q_{1,2}$, this trajectory cannot return to D_1 through any point in D_2 . Since the “graph of irreducible components” associated to E contains no loop, we conclude that l^+ will never return to D_1 . In other words, if $l^+ \subset E$, then l^+ will eventually be “captured” by an irreducible component of E that can be supposed to be D_1 .

On the other hand, the trajectory l^+ cannot approach any singularity $q \in D_1$ where the leaves of \mathcal{H} restricted to D_1 approach q . Otherwise l^+ would leave D_1 by means of the separatrix of \mathcal{F} at q which is transverse to D_1 . This, in fact, implies that the complement of a compact part of l^+ does not accumulate on any singularity. Since D_1 is compact and l^+ is of infinite length, Theorem 5.7 can be employed to ensure that the holonomy group of $D_1 \setminus \text{Sing}(\mathcal{F})$, w.r.t. the foliation \mathcal{F} , contains a hyperbolic element. This is however impossible since it would force T to be concentrated over D_1 . \square

Recalling Remark (6.6), we can suppose that l^+ arrive to E through p . In other words, around p there is a separatrix of \mathcal{F} transverse to E (given by $\{v = 0\}$ and denoted by S_p) in the local coordinates (u, v) used in the proof of Lemma (6.5), with $\{u = 0\} \subset E$. Since E is a connected component of the set of all compact curves invariant by \mathcal{F} that are contained in \mathcal{K} , it follows that this separatrix is contained neither in the divisor of zeros and poles of ω nor in $(\omega_1)_0$. Similarly, in coordinates (w, t) around q , $\{w = 0\} \subset E$, there is a separatrix of \mathcal{F} at q which is transverse to E (given by $\{t = 0\}$ and denoted by S_q). Again this separatrix is not contained in $(\omega)_0 \cup (\omega)_\infty \cup (\omega_1)_0$.

To prove Proposition 6.4 let us suppose for a contradiction the existence of a \mathcal{H} -trajectory l^+ in \mathcal{K}^0 satisfying the above conditions but which is not a closed trajectory passing through singular points of \mathcal{F} . Recall that l^+ accumulates on itself, i.e. it has non-trivial recurrence. By

using the local coordinates (u, v) , (w, t) introduced above, the non-trivial recurrence of l^+ implies the existence of points $(u_n, v_n) = (e^{2\pi i \theta_n}, v_n)$ satisfying the following:

1. $(u_n, v_n) = (e^{2\pi i \theta_n}, v_n)$ belongs to l^+ for every $n \in \mathbb{N}$.
2. Both sequences $\{\theta_n\}$, $\{v_n\}$ converge to zero.

Actually we can be more precise. Let U_E be a small “tubular” neighborhood of E . Let us consider the “full” sequence of “first returns” of l^+ to U_p which will still be denoted by $(e^{2\pi i \theta_n}, v_n)$. If U_E is appropriately chosen, then the local connected component of l^+ through $(e^{2\pi i \theta_n}, v_n)$ satisfies the conclusions of Proposition (3.4). We have:

Claim: We can assume that $\theta_n = 0$ for every n .

The above assumption is not really needed from a strict point of view. It simply allows us to shorten our discussion which applies equally well to the general case. It can also be formalized by again locally deforming the leaves of \mathcal{H} on a neighborhood of the circle $(e^{2\pi i \theta}, 0)$, $\theta \in [0, 2\pi)$. This deformation is essentially given by the local holonomy of $\{v = 0\}$ and does not affect neither the global dynamics of l^+ nor the estimates involved in the holonomy of (compact pieces of) l^+ .

Summarizing, in what follows we assume that l^+ enters the “tubular neighborhood” U_E by means of a sequence of points having the form $P_n = (1, v_n)$. Besides the sequence $\{|v_n|\}$ converges to $0 \in \mathbb{C}$. In particular, these points belong to Σ_{in} , a local transverse section through the point $(1, 0)$ in (u, v) -coordinates. In the sequel we sometimes identify the point $(1, v) \in \Sigma_{\text{in}}$ with the point $v \in \mathbb{C}$, thus identifying Σ_{in} itself with a neighborhood of $0 \in \mathbb{C}$.

Let us briefly review what is the nature of the holonomy map of \mathcal{F} associated to a trajectory of \mathcal{H} as above. We begin with a definition on U_E involving the generalized Dulac transform introduced in Section 4. It is clear that the segment of l^+ delimited by the points p, q above verifies the condition discussed in Section 4 in connection with the generalized Dulac transform. Let then Σ_{out} be a transverse section through the point $(1, 0)$ in (w, t) -coordinates (on a neighborhood of q) so that the corresponding generalized Dulac transform is well-defined. The last statement can be made precise as follows: Let $V_0 \subset \Sigma_p$ be a simply connected domain containing a point $(1, z) \in \mathcal{K}^0 \cap \Sigma_{\text{in}}$. According to Sections 3, 4, the oriented trajectory $l_{(1, z)}$ of \mathcal{H} through $(1, z)$ intersects Σ_{out} at a point $(z', 1) \in \mathcal{K}^0 \cap \Sigma_q$. Parameterizing by $c : [0, 1] \rightarrow l_{(1, z)}$ the segment of $l_{(1, z)}$ delimited by $(1, z)$ and $(z', 1)$, we ask the generalized Dulac transform $\text{GDul} : V_0 \rightarrow \Sigma_{\text{out}}$ to be well-defined w.r.t. the path c (in the sense of Sections 3, 4). Obviously $\text{Dul}(1, z) = (z', 1)$. Modulo reducing U_E , we can suppose that U_E is the saturation of Σ_{out} by \mathcal{F} . Finally let λ denote the exponent associated with this generalized Dulac transform as in the context of Proposition 4.6.

Now consider the compact set $\mathcal{K}^0 \setminus U_E$ where the foliation \mathcal{F} is regular. Here the holonomy associated to \mathcal{F} and to a segment of the trajectory l^+ contained in $\mathcal{K}^0 \setminus U_E$ has a clear meaning. Besides if $c : [0, 1] \rightarrow \mathcal{K}^0 \setminus U_E$ is a path parameterizing a segment of l^+ , then $\text{Hol}(c)$ satisfies the conclusions of Theorem 5.7, in particular $\text{Hol}(c)$ is defined on a transverse disc of uniform radius $\delta > 0$ (regardless of the length of c).

Resuming the notations of Proposition (4.6), there are two cases to be considered according to whether or not the value of λ is rational.

- Let us first suppose that $\lambda \in \mathbb{R} \setminus \mathbb{Q}$.

Recall that λ is less than or equal to 1. Let $l_{p, q}^+$ denote the segment of l^+ delimited by p, q . By choosing a point in this segment, we can consider the holonomy of \mathcal{F} generated by

the local holonomy maps associated to the separatrices of the singularities of \mathcal{F} lying in $l_{p,q}^+$ (in particular p, q). This group is Abelian. Otherwise, a non-trivial commutator in this group would be “parabolic” in the sense that it is tangent to the identity. As in Lemma (4.3), the existence of this local diffeomorphism would yield a contradiction since $l^+ \subset \mathcal{K}$ and T is diffuse having \mathcal{K} as its support.

Clearly the local holonomy maps associated to the above mentioned singularities can be identified with elements of $\text{Diff}(\mathbb{C}, 0)$ by appropriately choosing local transverse section and (ordinary) Dulac transforms. The fact that $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ then implies the existence of a holonomy map h in this group that, under the above identification, is an element of $\text{Diff}(\mathbb{C}, 0)$ whose linear part is an irrational rotation. Then the following elementary statement holds:

FACT 1: Given an arbitrary $l \in \mathbb{N}^*$, there is a constant $C = C(l, 2\pi\lambda)$ such that, for every $r \in \mathbb{R}_+$ sufficiently small and $z \in \mathbb{C}$ with $|z| = r$, the sets $B_z(C.r), h(B_z(C.r)), \dots, h^{l-1}(B_z(C.r))$ are pairwise disjoint (where $B_z(C.r)$ stands for the ball about z of radius $C.r$).

Now consider a strictly monotone sequence $\{r_j\} \subset \mathbb{R}_+$ converging to zero. For each j , denote by $B(r_j) \subset \Sigma_{\text{in}}$ the ball of radius r_j about $0 \simeq (1, 0) \in \Sigma_{\text{in}}$. Next recall that $(1, v_n) \in \Sigma$ is the sequence of the “first returns” of l^+ to U_E .

Proof of Proposition 6.4 when $\lambda \in \mathbb{R} \setminus \mathbb{Q}$: We begin by fixing $l \in \mathbb{N}$ larger than $2\pi/\lambda$. Next we consider a constant $C = C(l, 2\pi\lambda)$ as in Fact 1. Finally let us denote by μ, ν suitable measures representing the current T on the transverse sections $\Sigma_{\text{in}}, \Sigma_{\text{out}}$ respectively.

Consider a ball $B(r) \subset \Sigma_{\text{in}}$ with r very small. Let $V_\alpha(r) \subset B_r \subset \Sigma_p$ be a sector of angle $\alpha < 2\pi\lambda$ and radius r_j . The invariance of μ under the local diffeomorphism h obtained as the holonomy map of $\{v = 0\}$ implies that, for every $\epsilon > 0$ fixed, one has

$$\frac{\mu(V_\alpha(r))}{\mu(B(r))} \geq (1 - \epsilon)\lambda \quad (22)$$

provided that r is very small and that α is sufficiently close to $2\pi\lambda$.

Now let us choose $j_0 \in \mathbb{N}$ very large and denote by v_{n_0} the smallest positive integer n such that $v_n \in B(r_{j_0})$. Set $r_0 = |v_{n_0}|$ and denote by $B(r_0)$ (resp. $B(2r_0)$) the ball of radius r_0 (resp. $2r_0$) about $0 \simeq (1, 0) \in \Sigma_{\text{in}}$. For α very close to $2\pi\lambda$, let us denote by $V_\alpha(2r_0)$ the sector of angle α and radius $2r_0$ which is divided into two equal parts by the semi-line joining 0 to v_n . Modulo taking r_0 sufficiently small (i.e. j_0 large enough), we can consider a generalized Dulac transform GDul which is well-defined on $V_\alpha(2r_0)$ (w.r.t. some path c fixed once and for all).

The image $W(2r_0)$ of $V_\alpha(2r_0)$ under GDul is contained in a disk $B'(r'_0) \subset \Sigma_{\text{out}}$ of radius $r'_0 \simeq r_0^{1/\lambda}$ (cf. Proposition 4.6). Again by choosing r_0 small enough, this estimate yields $r'_0 < 2Cr_0$ where C is the constant fixed above. Also we know that $\nu(W(2r_0)) > (1 - \epsilon)\mu(V_\alpha(2r_0))$.

Finally let $c : [0, 1] \rightarrow \mathcal{K}^0$ be a parametrization of the segment of trajectory l^+ going from the first point in which l^+ intersects Σ_q to the point $(1, v_{n_0}) \in \Sigma$. Note that c remains away from the neighborhood U_E . In fact, its distance to U_E is bounded below by a uniform constant times r_0 . In particular, for r_0 small enough, this implies that the holonomy map $\text{Hol}(c(t))$ is well-defined on $W(2r_0)$ for every $t \in [0, 1]$. Besides Theorem 5.7 also ensures that $\text{Hol}(c)(W(2r_0))$ is contained in a disc of radius less than $2r'_0$ about $v_{n_0} \simeq (1, v_{n_0}) \in \Sigma$. Since $|v_{n_0}| = r_0$, this ensures that $\text{Hol}(c)(W(2r_0)) \subset B(2r_0)$. Furthermore we have $\mu[\text{Hol}(c)(W(2r_0))] = \nu(W(2r_0))$ since μ, ν are local transverse measures representing the invariant current T . Thanks to Estimate (22), it follows that

$$\mu[\text{Hol}(c)(W(2r_0))] \geq (1 - \epsilon)\lambda\mu(B(2r_0)). \quad (23)$$

Finally, since $r'_0 < 2Cr_0$, the set $\text{Hol}(c)(W(2r_0))$ has l images pairwise disjoint under the holonomy map h . Since all these images have the same μ measure, the total measure of their union is $l \cdot \mu[\text{Hol}(c)(W(2r_0))] > \mu(B(2r_0))$ in view of the choice of l and for ϵ very small. This yields the desired contradiction since the union of these images is contained in $B(2r_0)$. Therefore our statement is proved in the present case. \square

• Let us now suppose that $\lambda \in \mathbb{Q}$.

This case is somehow similar to the preceding one. The main difference is that Fact 1 no longer holds. On the other hand, we know that \mathcal{F} is linearizable around every singularity in $l_{p,q}^+$ (Proposition 4.1). Yet the local holonomy maps associated to these singularities generate an Abelian group which is therefore finite and hence conjugate to a (finite) group of rotations. This will allow us to make precise asymptotic calculations so as to dispense with the “ ϵ margin” involved in preceding discussion. In fact, standard arguments involving this group and the nature of the singularities of \mathcal{F} contained in E shows that the restriction of \mathcal{F} to U_E admits a non-constant holomorphic first integral (see [M-M], [Pa]). The existence of this integral however will not be necessary in what follows.

Let us resume the notations of the case where λ was not rational. Since the above mentioned holonomy group associated to the curve L is conjugate to a finite group of rotations, we can find a local diffeomorphism h in this group which is itself a rotation of angle $2\pi\lambda$. Let $V_{2\pi\lambda}(r) \subset \Sigma_{\text{in}}$ be a sector of angle $2\pi\lambda$ and radius r . If $B(r) \subset \Sigma_p$ is the corresponding ball of radius r , we obviously have $\mu[V_{2\pi\lambda/\lambda_1}(r)] = \lambda\mu(B(r))$.

Proof of Proposition 6.4 when $\lambda \in \mathbb{Q}$: Again choose $j_0 \in \mathbb{N}$ very large and denote by v_{n_0} the smallest positive integer n such that $v_n \in B(r_{j_0})$. Set $r_0 = |v_{n_0}|$. Next let $V_{2\pi\lambda}(2r_0)$ be the sector of angle $2\pi\lambda$ and radius $2r_0$ which is divided into two equal parts by the semi-line joining 0 to v_n . Modulo taking r_0 sufficiently small (i.e. j_0 large enough), we can consider a generalized Dulac transform GDul which is well-defined on $V_{2\pi\lambda}(2r_0)$ (w.r.t. some path c fixed once and for all). The image $W(2r_0)$ of $V_{2\pi\lambda}(2r_0)$ under GDul is contained in a disc $B'(r'_0) \subset \Sigma_{\text{out}}$ of radius $r'_0 \simeq r_0^{1/\lambda}$ (cf. Proposition 4.6). We note that the possibility of having $\lambda = 1$ is not *a priori* excluded so that we first suppose $\lambda < 1$. As in the proof of the case “ λ irrational”, the above mentioned disc is taken by the holonomy of l^+ to a set $\text{Hol}(c)(W(2r_0)) \subset B(2r_0) \subset \Sigma_{\text{in}}$. Furthermore we have

$$\lambda\mu(B(2r_0)) = \mu[V_{2\pi\lambda}(2r_0)] = \nu[W(2r_0)] = \mu[\text{Hol}(c)(W(2r_0))].$$

However the set $\text{Hol}(c)(W(2r_0))$ has the “denominator of $\lambda \in \mathbb{Q}$ ” images pairwise disjoint under the iterations of the rational rotation h of angle $2\pi\lambda$. Since they are all contained in $B(2r_0)$, it follows that λ is the inverse of an integer. In this case $\mu[\bigcup_{i=1}^{1/\lambda} h^i(\text{Hol}(c)(W(2r_0)))] = \mu(B(2r_0))$. Finally because $0 \simeq (1, 0) \in \Sigma_{\text{in}}$ is not contained in the closure of the set $\bigcup_{i=1}^{\lambda_1} h^i(\text{Hol}(c)(W(2r_0)))$, it follows that a sufficiently small neighborhood of $0 \in \Sigma_{\text{in}}$ has μ -measure zero. Since \mathcal{K}^0 is in the support of $[T]$, it must not intersect the neighborhood in question. This yields the desired contradiction since \mathcal{K}^0 accumulates on p .

Finally if we have $\lambda = 1$, then GDul yields an identification of neighborhoods of the origin in $\Sigma_{\text{in}}, \Sigma_{\text{out}}$ which takes μ to ν . In other words, the segments of l^+ passing through U_E behaves as they remained “away from the singular set of \mathcal{F} ”. In this case the conclusion follows simply from the contractive behavior of the holonomy of \mathcal{F} defined with the help of the segments of l^+ contained in the complement of U_E . The proof of Proposition 6.4 in the case corresponding to the connectedness of E is now over. \square

Proof of Proposition 6.4 in the general case: To finish this section, let us now show how the previous arguments can naturally be adapted to yield the proof of Proposition 6.4 when E contains more than one connected component. Suppose first that, instead of a single connected component E , there are two connected components E_1, E_2 . We consider a trajectory l^+ of \mathcal{H} arriving to E_1 through a singularity p_1 and leaving E_1 through another singularity q_1 (as in Lemmas 6.5 and 6.7). The first possibility that may occur is a “saddle-connection” between q_1 and a singularity $p_2 \in E_2$. More precisely, it may happen that the separatrix of \mathcal{F} at q_1 which is transverse to E_1 coincides with a separatrix of another singularity $p_2 \in E_2$ of \mathcal{F} . Then l^+ will arrive to E_2 through p_2 .

The solution of this first difficulty is provided by the proof of Proposition 4.6. In fact, by using the leaf of \mathcal{F} joining q_1 to p_2 , we can define a new “generalized Dulac transform” encompassing both E_1, E_2 as they were a single connected component. This definition is straightforward and the proof of Proposition 4.6 shows that the resulting “Dulac transform” still satisfies the conditions given in the statement in question.

Suppose now that there is no “saddle-connection” in the sense described above between E_1, E_2 . The difficulty here arises from the fact that l^+ may accumulate on E_2 before returning to E_1 . Again we suppose that l^+ arrives to E_1 (resp. E_2) through a singularity p_1 (resp. p_2) and leaves it through a singularity q_1 (resp. q_2). Let S_{p_2} denote the separatrix of \mathcal{F} at p_2 which is transverse to E_2 .

Again we keep the notations used in the course of this section. Let us then consider the image $W(2r_0)$ of $V_\alpha(2r_0)$ under the generalized Dulac transform associated to E_1 , GDul_1 . As l^+ continues from q_1 to p_2 , we let $\text{Hol}(W(2r_0))$ denote the image of $W(2r_0)$ by the corresponding holonomy map. The special difficulty here is that, when approaching p_2 , l^+ may become “very close” to S_{p_2} . In particular, with the obvious identifications, it may happen that $\text{Hol}(W(2r_0))$ contains S_{p_2} . This situation prevents us from considering the “Dulac transform” associated to E_2 , GDul_2 , as being defined on all of $\text{Hol}(W(2r_0))$. To deal with this case, we proceed as follows. First we note that, in the hard case, this phenomenon should occur for “every sequence of returns” of l^+ to the transverse section in which GDul_1 is defined. Under this assumption, we substitute l^+ by a trajectory l_2^+ of \mathcal{H} which has properties analogues to those of l^+ and, furthermore, is contained in S_{p_2} . The existence of this trajectory is clear since l^+ accumulates on S_{p_2} . We then start our argument with l_2^+ so that the Dulac transform associated to E_2 is automatically well-defined. We claim that, for l_2^+ , the Dulac transform associated to E_1 will also be well-defined on the appropriate domain. It is clear that the desired statement results easily from this claim.

To check the claim, we note that l^+ is close to l_2^+ at the same order of the diameter of $W(2r_0)$ near p_2 . In turn, this diameter is small when compared to r_0 (cf. Proposition 4.6). If l^+, l_2^+ remain close to each other for all time, then l_2^+ reaches the domain of definition of GDul_1 at a point “very close” to a point of return of l^+ to this domain (i.e. the distance between these two points is small to order superior to the distance of the return point to E_1). Since the diameter of the corresponding $W(2r_0)$, $\text{Hol}(W(2r_0))$ is also small, the claim follows at once. This therefore completes the argument modulo the assumption that l^+, l_2^+ remain close to each other for all time. This assumption however can always be made. In fact, we first observe that the leaf L_2 of \mathcal{F} containing l_2^+ approaches the trajectory l^+ due to the contracting behavior of the holonomy along l^+ . By a simple argument of continuous dependence for solutions of differential equations, l^+, l_2^+ remain close for an *a priori* fixed period of time. But at the end of this period of time, we can modify l_2^+ inside L_2 by adding a short line (transverse to \mathcal{H}) so as to bring the modified

trajectory l_2^+ close again to l^+ . This new trajectory l_2^+ satisfies all the previous requirements and establishes the claim.

Now it is clear that the existence of several E_1, E_2, \dots connected components does not pose any new intrinsic difficulty. The proof of Proposition 6.4 is finally completed. \square

7 Proofs for the main results

To be able to prove the theorems stated in the Introduction, we shall need to consider the closed trajectory l^+ whose existence is ensured by Proposition 6.4. This trajectory will be referred to as a *singular* closed trajectory since it passes through the singularities of \mathcal{F} . In what follows, we shall keep the notations and the terminology of the preceding section.

Denote by L_0, L_1, \dots, L_m the leaves of \mathcal{F} that contain a non-trivial segment of l^+ . For $i = 0, \dots, m-1$, L_i intersects L_{i+1} at a singularity of \mathcal{F} belonging to the Siegel domain. Also L_k intersects L_0 at a Siegel singularity so that the leaves L_0, L_1, \dots, L_m form a loop by means of their saddle-connexions.

Fix a base point $p \in l^+ \cap L_0$ and consider a local transverse section Σ at p . To prove our main results we are going to consider the pseudogroup Γ of transformations of Σ obtained by the collection of first return maps over paths contained in the leaves L_0, L_1, \dots, L_m . More generally denote by \mathcal{L} the union of the (finitely many) leaves of \mathcal{F} defined by the following rules:

- L_0, L_1, \dots, L_m belongs to \mathcal{L} .
- If L belongs to \mathcal{L} and L (locally) defines a separatrix for a Siegel singularity p of \mathcal{F} , then the global leaf obtained from the other separatrix of \mathcal{F} at p must also belong to \mathcal{L} .

The pseudogroup Γ is then obtained by means of first return maps defined over all paths contained in \mathcal{L} .

One first element f of Γ corresponds of course to the singular loop l^+ . As already seen, we can suppose that f is a (ramified) map of the form $f(z) = z^\lambda(1+u(z))$ where $\lambda > 1$ and $u(0) = 0$ (u being defined on a neighborhood of $0 \in \mathbb{C}$). Note that λ need not be an integer so that f should be thought of as a “ramified” map. Yet, in sectors of angle slightly smaller than $2\pi/\lambda$, f is well-defined and one-to-one onto its image.

Another element of Γ , denoted by g , corresponds to the local holonomy map arising from the singularities of L_0, \dots, L_m . Observe that at least one of the Siegel singularities of \mathcal{F} lying in l^+ must have eigenvalues different from $1, -1$. In fact, otherwise f would have no ramification and thus it would consist of a hyperbolic contraction defined on a neighborhood of $p \in \Sigma$. Therefore every invariant measure on Σ would automatically be concentrated at p what it is impossible. Summarizing, we conclude that Γ contains an element g defined about $p \simeq 0 \in \mathbb{C}$ and having the form

$$g(z) = e^{2\pi i \alpha} z + \text{h.o.t.}$$

with $\alpha \in (0, 1)$. This leads us to study the dynamics of a pseudogroup Γ containing elements f, g as above.

Recall that $\lambda > 1$. If λ were an integer, then the classical theorem of Böttcher would provide coordinates where $f(z) = z^\lambda$. However, in general, the map f is ramified so that it is well-defined

only on suitable sectors. The “position” of the sectors where we want to define a particular determination of f are naturally permuted by means of the local holonomy maps associated to the Siegel singularities of \mathcal{F} lying in l^+ . Indeed, the ambiguity in the definition of f as compositions of ordinary holonomy maps and suitable Dulac transforms is precisely codified by the local holonomy maps arising from the mentioned singularities. In particular two different determinations of f commute in the obvious sense with the corresponding local holonomy maps. These elementary facts will freely be used in what follows.

Now, even though λ is not an integer, the method of Böttcher still provides a conjugacy between f and $z \mapsto z^\lambda$ over appropriate sectors. Whereas the conjugacy map is clearly not defined about $0 \in \mathbb{C}$, it has all natural asymptotic properties at $0 \in \mathbb{C}$. By using one of these coordinates, we can suppose that Γ contains the map $f(z) = z^\lambda$ along with at least one map g of the form $g(z) = e^{2\pi i \alpha} z + r(z)$ where $\alpha \in (0, 1)$ and $\|r(z)\| \leq C\|z\|^2$ for some constant C . Furthermore different determinations of f are naturally permuted by g .

Before continuing let us make some elementary remarks about the function “ k^{th} -root”. More precisely let $k \in \mathbb{R}$, $k > 1$, be fixed. Consider the map $z \mapsto (1+z)^k$ which is well-defined for $\|z\| < 1/2$. The corresponding derivative is simply $(1+z)^{(1-k)/k}/k$. In particular, for $\|z\| < 1/2$, the norm of its derivative is uniformly bounded by

$$\frac{1}{k} 2^{(k-1)/k} \leq \frac{2}{k}. \quad (24)$$

Next consider the element h_1 of Γ defined by $h_1(z) = f^{-1} \circ g \circ f(z)$ and note that h_1 is well-defined on a uniform sector (slightly smaller than the sector in which f was defined). More generally different determinations of h_1 are naturally permuted by g since so are the determinations of f . Similarly we define

$$h_n(z) = f^{-n} \circ g \circ f^n(z).$$

Our first task is to show that the elements h_n are defined on a uniform domain and that they converge to the identity on this domain. For this let us set $g(z) = e^{2\pi i \alpha} z + c_2 z^2 + c_3 z^3 + \dots$. Now note that h_1 admits the form

$$h_1(z) = e^{2\pi i \alpha} z (1 + c_2 z^\lambda + c_3 z^{2\lambda} + \dots)^{1/\lambda}.$$

The expression $c_2 z^\lambda + c_3 z^{2\lambda} + \dots = r(z^\lambda)/z^\lambda$ is clearly less than $1/2$ for $\|z\|$ sufficiently small. In particular $h_1(z)$ is actually holomorphic on a neighborhood of $0 \in \mathbb{C}$. In addition, Estimate 24 yields

$$\|h_1(z) - e^{2\pi i \alpha / \lambda} z\| \leq \frac{2}{\lambda} C \|z\|^\lambda \quad (25)$$

on the same domain. A direct inspection shows that $h_n(z)$ is holomorphic on the same neighborhood of $0 \in \mathbb{C}$ and that it satisfies the following estimate

$$\|h_n(z) - e^{2\pi i \alpha / \lambda^n} z\| \leq \frac{2}{\lambda^n} C \|z\|^{\lambda^{2n-1}}. \quad (26)$$

Because $\lambda > 1$, we obtain

Lemma 7.1 *For $\tau > 0$ sufficiently small, all the h_n are holomorphic and well-defined on the disc $B_\tau(0)$ of radius τ about $0 \in \mathbb{C}$. Furthermore these diffeomorphisms converge uniformly to the identity on $B_\tau(0)$.* \square

Recall that a vector field Y defined on a neighborhood U of $0 \in \mathbb{C}$ is said to *belong to the closure of Γ (relative to U)* if for every $V \subset U \subset \mathbb{C}$ and $t_0 \in \mathbb{R}_+$ so that $\phi_Y^t(V)$ is well-defined for all $0 \leq t \leq t_0$, the map $\phi_Y^{t_0} : U \rightarrow \phi_Y^{t_0}(U) \subset U$ is a uniform limit of elements of Γ defined on V . Here ϕ_Y^t stands for the local flow generated by Y . From the definition it follows that $\phi_Y^{t_0}$ is holomorphic as a uniform limit of holomorphic maps (contained in Γ). Next we have:

Proposition 7.2 *The vector field whose local flow consists of rotations about $0 \in \mathbb{C}$ belongs to the closure of Γ (relative to the disc $B_{\tau/2}(0)$). In other words, every rotation $R_\beta : z \mapsto e^{2\pi i \beta} z$ is a uniform limit on $B_{\tau/2}(0)$ of actual elements of Γ .*

Proof. Fix a rotation $R_\beta(z) = e^{2\pi i \beta} z$. We need to find a sequence of elements in Γ that approximate R_β on $B_{\tau/2}(0)$. This sequence can explicitly be obtained as follows. For n large enough let k_n be the integral part of $\beta \lambda^n / \alpha$. Clearly the linear part of h_n at $0 \in \mathbb{C}$ is a rotation of angle $[\beta \lambda^n / \alpha] \alpha / \lambda^n = k_n \alpha / \lambda^n$. In particular the difference $|\beta - k_n \alpha / \lambda^n|$ is bounded by α / λ^n which, in turn, tends to zero when $n \rightarrow \infty$ (since $\lambda > 1$). Therefore to establish the proposition it suffices to check that the sequence $\{h_n^{k_n}\}_{n \in \mathbb{N}} \subset \Gamma$ satisfies the two conditions below.

1. For n very large, $h_n^{k_n}$ is well-defined on $B_{\tau/2}(0)$.
2. On $B_{\tau/2}(0)$, $h_n^{k_n}$ converges uniformly towards its own linear part at $0 \in \mathbb{C}$.

These conditions will simultaneously be verified as consequences of Estimate (26). To abridge notations, denote by R_n the rotation of angle α / λ^n about $0 \in \mathbb{C}$. The linear character of R_n gives $D_z R_n = R_n$ for every $z \in \mathbb{C}$. In particular the norm $\|D_z R_n\|$ is constant equal to 1. Next observe that, for $\|z\|$ sufficiently small, Estimate (26) yields

$$\begin{aligned}
\|h_n^2(z) - R_n^2(z)\| &= \|h_n^2(z) - R_n \circ h_n(z) + R_n \circ h_n(z) - R_n^2(z)\| \\
&\leq \|(h_n - R_n) \circ h_n(z)\| + \|R_n(h_n(z)) - R_n(R_n(z))\| \\
&\leq \frac{2C}{\lambda^n} \|h_n(z)\|^{\lambda^{2n-1}} + \sup_{B_{\tau/2}(0)} \|DR_n\| \cdot \|h_n(z) - R_n(z)\| \\
&\leq \frac{2C}{\lambda^n} \|h_n(z)\|^{\lambda^{2n-1}} + \frac{2C}{\lambda^n} \|z\|^{\lambda^{2n-1}} \\
&= \frac{2C}{\lambda^n} (\|h_n(z)\|^{\lambda^{2n-1}} + \|z\|^{\lambda^{2n-1}}).
\end{aligned}$$

If we set $\|h_n^3(z) - R_n^3(z)\| \leq \|h_n(h_n^2(z)) - R_n(h_n^2(z))\| + \|R_n(h_n^2(z)) - R_n(R_n^2(z))\|$ and repeat the above procedure, it follows that

$$\|h_n^3(z) - R_n^3(z)\| \leq \frac{2C}{\lambda^n} \|h_n^2\|^{\lambda^{2n-1}} + \frac{2C}{\lambda^n} (\|h_n(z)\|^{\lambda^{2n-1}} + \|z\|^{\lambda^{2n-1}}).$$

By induction, if l is such that all the iterates $h_n(z), h_n^2(z), \dots, h_n^{l-1}(z)$ remain in the disc of radius τ for every z with $\|z\| < \tau/2$, we derive the following estimate

$$\|h_n^l(z) - R_n^l(z)\| \leq \frac{2C}{\lambda^n} \left(\|z\|^{\lambda^{2n-1}} + \|h_n(z)\|^{\lambda^{2n-1}} + \dots + \|h_n^{l-1}(z)\|^{\lambda^{2n-1}} \right) \quad (27)$$

$$\leq \frac{2lC}{\lambda^n} \|\tau\|^{\lambda^{2n-1}}. \quad (28)$$

For $\tau > 0$ small and fixed, we see that $l > k_n$. In fact, since $k_n < \beta\lambda^n/\alpha$, we obtain for n sufficiently large

$$\|h_n^{k_n}(z) - R_n^{k_n}(z)\| \leq \frac{2\beta C}{\alpha} \|\tau\|^{\lambda^{2n-1}} \leq \frac{\tau}{2}.$$

Furthermore $\|R_n^{k_n}(z)\| < \tau/2$ provided that $\|z\| < \tau/2$. This remark combines with the preceding estimate to guarantee that $h_n^{k_n}$ is well-defined on $B_{\tau/2}(0)$ for n large as above. Since the right hand side in (28) tends to zero as $n \rightarrow \infty$ ($\lambda > 1$), we can also conclude that $\{h_n^{k_n}\}$ converges uniformly towards $R_n^{k_n}$ on $B_{\tau/2}(0)$. This finishes the proof of the proposition. \square

Let us denote by $\bar{\Gamma}$ the closure of Γ (relative to $B_{\tau/2}(0)$). Naturally the contents of Proposition (7.2) is that all the rotations $R_\beta = e^{2\pi i\beta}z$ belong to $\bar{\Gamma}$. With this information in hand, let us go back to our original setting where Γ is supposed to preserve a measure μ on $B_{\tau/2}(0)$ which, in addition, has no Dirac components. It is immediate to check that μ must also be preserved by all elements lying in $\bar{\Gamma}$. In particular μ is preserved by the group of rotations $z \mapsto e^{2\pi i\beta}z$.

Consider polar coordinates r, θ for $B_{\tau/2}(0)$. Since the only measures on the circle that are preserved by the group of rotations are the constant multiples of the Haar measure, Fubini's theorem provides:

Lemma 7.3 *The measure μ is given in polar coordinates by $T(r)drd\theta$ where T is naturally identified with a 1-dimensional distribution.* \square

Clearly all measures μ having the form indicated in the preceding lemma are automatically invariant by the group of rotations. The fact that μ is also preserved by $f(z) = z^\lambda$ can then be translated into the functional equation

$$T(r) = \lambda^2 r^{\lambda-1} T(r^\lambda). \quad (29)$$

We are now able to prove Theorem A.

Proof of Theorem A. Recall that l^+ is contained in a closed set \mathcal{K} that is minimal for \mathcal{F} . Let us point out that the condition of having \mathcal{K} minimal has not been used so far. This condition however is going to play a role in the sequel. To prove the theorem we are going to show that \mathcal{K} is itself an algebraic curve. To do this consider the collection \mathcal{L} of leaves of \mathcal{F} as defined in the beginning of this section. Clearly we have $l^+ \subset \mathcal{L}$. Next denote by $\bar{\mathcal{L}}$ the closure of \mathcal{L} and consider the dimension of the set $\bar{\mathcal{L}} \setminus \mathcal{L}$ of the proper accumulation points of \mathcal{L} . According to the classical Remmert-Stein theorem, if the codimension of $\bar{\mathcal{L}} \setminus \mathcal{L}$ is at least two, then $\bar{\mathcal{L}}$ is itself an analytic set so that the statement follows at once.

Thus we can suppose that the codimension of $\bar{\mathcal{L}} \setminus \mathcal{L}$ is strictly less than two. In particular $\bar{\mathcal{L}} \setminus \mathcal{L}$ cannot be contained in the singular set of \mathcal{F} . Thus we can consider a point $p \in \bar{\mathcal{L}} \setminus \mathcal{L}$ that is regular for \mathcal{F} . By considering a plaque of \mathcal{F} containing p , we see that \mathcal{L} must non-trivially accumulate on this plaque. Since \mathcal{K} is minimal, it then follows that \mathcal{L} has non-trivial recurrence. In other words, on Σ (identified with the disc $B_\tau(0) \subset \mathbb{C}$), we can consider a point $q \in B_{\tau/2}(0)$, $q \neq 0$, belonging to \mathcal{L} .

To establish the statement we shall derive a contradiction from the preceding with the fact that q belongs to the support of the (transverse) invariant measure μ . To do this, consider the pseudogroup Γ' of first return maps defined over paths in \mathcal{L} but based at q . The group Γ' is conjugate to the group Γ . The desired contradiction arises as follows. Recall that the structure

of μ on $B_{\tau/2}(0)$ was already clarified by Lemma 7.3 and Equation (29). Because Γ' is conjugate to Γ , the analogous conclusions have to apply to a neighborhood of q as well. In particular μ is “constant” over suitable closed loops about q . Since μ is also “constant” over the initial circles about $0 \in B_{\tau/2}(0) \subset \mathbb{C}$, it follows that μ should be “constant” on a neighborhood of q , i.e. on a neighborhood of q the measure μ must be a constant multiple of the Lebesgue measure. This however contradicts the analogous of Equation (29) corresponding to the point q . The theorem is proved. \square

Let us close this paper with the proof of Theorem B. The method employed here is to large extent borrowed from [Pa] to which we refer for further details. The prototype of an equation admitting a Liouvillean first integral (integrable in the sense of Liouville) is the 1-dimensional equation $y' = a(x)y + b(x)$ for which an explicit solution involving two integrals can be obtained. In the complex domain, these integrals are in general multivalued so that, loosely speaking, we can say that the equation admits a first integral that is “twice multivalued”. Let us make this notion precise.

Keeping the preceding notations, let us denote by \mathcal{K} the algebraic curve obtained from Theorem A. In particular \mathcal{K} coincides with $\overline{\mathcal{L}}$ where \mathcal{L} was defined in the beginning of the section. In the sequel consider meromorphic 1-forms η inducing \mathcal{F} but being defined only on a neighborhood of \mathcal{K} in M . So, unlike the previously used form ω , η need not be globally defined. Consider also a collection of local representatives $\{(U_a, \eta_a)\}$ for η on a neighborhood of \mathcal{K} . The compatibility condition among the local representatives η_a being given by the condition $\eta_a = u_{ab}\eta_b$ where $u_{ab} \in \mathcal{O}^*(U_a \cap U_b)$. A *holomorphic integrating factor* for \mathcal{F} on the mentioned neighborhood consists of a collection $\{g_a\}$ of holomorphic functions, $g_a = u_{ab}g_b$, vanishing on \mathcal{K} and verifying

$$d\left(\frac{\eta_a}{g_a}\right) = 0.$$

The conditions above ensure that the local forms η_a/g_a can be glued together to yield a closed meromorphic form defining \mathcal{F} on a neighborhood of \mathcal{K} . Therefore every primitive H of the latter global form produces a multivalued first integral for \mathcal{F} , in fact, one has $\eta_a \wedge dH = 0$ for all a .

A Liouvillean first integral for \mathcal{F} on a neighborhood of \mathcal{K} as above consists of going one step further into the preceding discussion. A natural definition taken from [Pa] is as follows. Consider the universal covering $\Pi : \mathcal{U} \rightarrow M \setminus \mathcal{K}$ of $M \setminus \mathcal{K}$. The sheaf $\mathcal{O}_{\mathcal{U}}$ induces a sheaf over M corresponding to its direct image by Π and by the natural inclusion $M \setminus \mathcal{K} \hookrightarrow M$. The restriction of the latter sheaf to \mathcal{K} is going to be denoted by $\tilde{\mathcal{O}}$. By construction an element belonging to the fiber of $\tilde{\mathcal{O}}$ over a point $q \in \mathcal{K}$ is represented by a holomorphic function on $\Pi^{-1}(V \setminus V \cap \mathcal{K})$ where V stands for a neighborhood of $q \in M$. The property of unique lift of functions through Π allows us to identify $\mathcal{O}_{\mathcal{K}}$ to a subset of $\tilde{\mathcal{O}}$ whose elements are, in addition, invariant by the local covering automorphisms. With analogous constructions, we also define over \mathcal{K} the sheaves (of germs of) multivalued vector fields/holomorphic forms.

Clearly the exterior differential d can naturally be lifted to all above mentioned sheaves. An element H of $\tilde{\mathcal{O}}(V)$ is said to be a *primitive* if df is a 1-form invariant by the local covering automorphisms and admitting a meromorphic extension to \mathcal{K} . Similarly $H \in \tilde{\mathcal{O}}$ is said to be an *exponential of primitive* if df/f is a 1-form invariant by the local covering automorphisms and admitting a meromorphic extension to \mathcal{K} . Let $\mathcal{S}^+(V)$ (resp. $\mathcal{S}^\times(V)$) be the additive (resp. multiplicative) subgroup of primitives (resp. exponential of primitive) of $\tilde{\mathcal{O}}(V)$. The *first Liouvillean extension* of $\mathcal{O}(V)$, denoted by $\mathcal{S}^1(V)$ is the subring of $\tilde{\mathcal{O}}(V)$ generated by $\mathcal{S}^+(V)$, $\mathcal{S}^\times(V)$. The resulting presheaf turns out to be a sheaf over \mathcal{K} . This construction can be continued by

induction to yield higher order Liouvillean extensions of $\mathcal{O}(V)$ but we shall not need those here (see [Pa]).

A Liouvillean (or 1-Liouvillean) integrating factor for \mathcal{F} on a neighborhood of \mathcal{K} consists of a collection $\{g_a\}$ of elements in $\mathcal{S}^1(U_a)$ such that

$$g_a = u_{ab}g_b \quad \text{and} \quad d\left(\frac{\eta_a}{g_a}\right) = 0.$$

A Liouvillean integrating factor is called *distinguished* if g_a belongs to $\mathcal{S}^\times(V)$, ie. if dg_a/g_a is a meromorphic closed form. By using this terminology we can state a slightly more accurate version of Theorem B.

Theorem 7.4 *Under the assumptions of Theorem B the foliation \mathcal{F} admits a distinguished integrating factor. More precisely \mathcal{F} is given by a (Liouvillean) meromorphic closed form of type dg_a/g_a where $g_a \in \mathcal{S}^\times(V)$.*

To prove Theorem 7.4 consider the local transverse Σ through $p \in l^+$ along with the pseudogroup Γ of first return maps over paths contained in \mathcal{L} (recalling that $\overline{\mathcal{L}} = \mathcal{K}$). Recall that Σ is endowed with a coordinate z in which the first return f over l^+ becomes $f(z) = z^\lambda$ on suitable sectors. The new ingredient leading to the proof of the mentioned theorem is the following proposition.

Proposition 7.5 *The vector field $\mathcal{X} = z\partial/\partial z$ is projectively invariant by Γ . In other words, if $h \in \Gamma$ then*

$$h^*\mathcal{X} = c_h\mathcal{X}$$

for a constant c_h and whenever both sides are defined.

To not interrupt the discussion we shall prove this proposition later. In the sequel we shall derive Theorem 7.4. To make the argument more transparent, suppose first that \mathcal{Y} were a vector field on Σ fully invariant by Γ , ie. satisfying $h^*\mathcal{Y} = \mathcal{Y}$ for every $h \in \Gamma$. If this vector field exists, then it induces a vector field (or rather a 1-parameter subgroup of automorphisms) on the leaf space of \mathcal{F} (restricted to a neighborhood of \mathcal{K} as it will always be the case in what follows). More precisely, on a neighborhood of \mathcal{K} consider the sheaf $\Theta_{M\mathcal{F}}$ consisting of germs of holomorphic vector fields tangent to \mathcal{K} and preserving the foliation \mathcal{F} . If $\mathcal{Z}(V)$ is an element of $\Theta_{M\mathcal{F}}$ then it verifies $L_{\mathcal{Z}(V)}\eta_a \wedge \eta_a = 0$, where $L_{\mathcal{Z}(V)}$ denotes the Lie derivative, as a consequence of the fact that $\mathcal{Z}(V)$ preserves \mathcal{F} . Similarly we denote by $\text{Tang}_{M\mathcal{F}}$ the subsheaf of $\Theta_{M\mathcal{F}}$ constituted by those germs of vector fields that are tangent to \mathcal{F} . The sheaf $\text{Symm}_{M\mathcal{F}}$ of “symmetries” of \mathcal{F} is then defined by means of the following exact sequence

$$0 \longrightarrow \text{Tang}_{M\mathcal{F}} \longrightarrow \Theta_{M\mathcal{F}} \longrightarrow \text{Symm}_{M\mathcal{F}} \longrightarrow 0.$$

With this terminology, it is clear that the vector field \mathcal{Y} defined on Σ and invariant by the holonomy of \mathcal{F} naturally induces a section of $\text{Symm}_{M\mathcal{F}}$ which is still denoted by \mathcal{Y} .

Next let a function g_a be defined on each open set U_a by the equation $g_a = \eta_a(\mathcal{Y})$. Because \mathcal{Y} is identified with a (global) section of $\text{Symm}_{M\mathcal{F}}$, it follows that $g_a = u_{ab}g_b$ so that the collection

$\{g_a\}$ forms a holomorphic integrating factor for \mathcal{F} . In fact, the condition $L_{Z(V)}\eta_a \wedge \eta_a = 0$ combines with the Cartan formula $L_{\mathcal{Y}} = di_{\mathcal{Y}} + i_{\mathcal{Y}}d$ to yield

$$d\left(\frac{\eta_a}{g_a}\right) = 0.$$

In other words, the collection of 1-forms $\{\eta_a/g_a\}$ defines a closed meromorphic form η defined on a neighborhood of the *regular part* of \mathcal{K} in M . However, the extension of this form to the singular points of \mathcal{F} lying in \mathcal{K} poses no difficulties since all these singularities must have two non-vanishing real eigenvalues as a consequence of the discussion in Section 4 (cf. also [Pa]). Finally a multivalued meromorphic first integral for \mathcal{F} can be obtained by means of the (multivalued) integral

$$\int \eta.$$

In particular it follows that the ambiguity in the definition of the mentioned first integral is precisely determined by the group of periods of η .

Proof of Theorem 7.4. Since a vector field \mathcal{Y} on Σ that is invariant by the pseudogroup Γ need not exist, we shall generalize the preceding discussion to the projectively invariant vector field \mathcal{X} whose existence is ensured by Proposition 7.5.

Unlike the previous discussion, we are now going to exploit the fact that \mathcal{F} is given by a globally defined meromorphic form ω . Thus, if we consider the collection of local forms $\{\eta_a\}$ obtained from the restrictions of ω , we conclude that the transition functions u_{ab} are all constant equal to 1. Now on each open set U_a we consider a projectively invariant vector field \mathcal{X}_a . Thanks to Proposition 7.5 this collection of vector fields can be chosen so that $\mathcal{X}_a = c_{ab}\mathcal{X}_b$ where all the (transition) functions c_{ab} are constant. Again on the collection of open sets $\{U_a\}$, we define the local functions $g_a = \eta(\mathcal{X}_a)$. Thus $g_a = c_{ab}g_b$.

Now on each U_a consider the (local) meromorphic 1-form $\Omega_{1a} = dg_a/g_a$ (with simple poles over \mathcal{K}). Since $g_a = c_{ab}g_b$ we conclude that $\Omega_{1a} = \Omega_{1b}$ so that these local forms glue together into a meromorphic form Ω_1 defined on a neighborhood of \mathcal{K} . It is straightforward to check that the following relations hold:

$$\eta_a \wedge \Omega_{1a} = d\eta_a \quad \text{and} \quad d\Omega_{1a} = 0.$$

In turn these relations are equivalent to the fact that the local functions $\{g_a\}$ provide a distinguished Liouvillean integrating factor for \mathcal{F} on a neighborhood of \mathcal{K} , since $u_{ab} = 1$ and $\Omega_{1a} = \Omega_{1b}$ cf. [Pa]. For the same reason mentioned above the extension of these factors to the singularities of \mathcal{F} lying in \mathcal{K} poses no additional difficulty. The theorem is proved. \square

The reader will note that the condition $d\omega = \omega \wedge \Omega_1$ means that the restriction of Ω_1 to the leaves of \mathcal{F} actually coincides with the foliated form ω_1 .

As before we can use the collection $\{g_a\}$ to obtain a multivalued meromorphic first integral for \mathcal{F} . For this we consider the (multivalued) integrating factor $h_a = \int (\eta_a/g_a)$. Since the monodromy of the collection $\{(U_a, g_a)\}$ amounts to multiplication by a constant ($g_a = c_{ab}g_b$), the same type of monodromy acts on the collection $\{h_a\}$ by affine transformations. Using these multivalued functions $\{h_a\}$ we can define a (further multivalued) first integral analogous to the one found in the first case discussed above. In slightly vague terms, the resulting first integral is given by

$$\int \frac{\omega}{\int \Omega_1}.$$

In particular the ambiguity in the definition of the first integral above lies in the period groups of ω and Ω_1 .

Let us now prove Proposition 7.5.

Proof of Proposition 7.5. Denote by $\phi_{\mathcal{X}}$ (resp. $\phi_{i\mathcal{X}}$) the real (resp. purely imaginary) flow generated by \mathcal{X} . Note that $\phi_{i\mathcal{X}}$ is the real 1-parameter group consisting of rotations about $0 \in \mathbb{C}$. Proposition 7.2 then says that $\phi_{i\mathcal{X}}$ is contained in $\bar{\Gamma}$. Next suppose for a contradiction that $h \in \Gamma$ does not preserve \mathcal{X} up to a multiplicative constant.

By construction Γ admits a finite generating set whose elements either are defined on a neighborhood of $0 \in \mathbb{C}$ or are defined on an appropriate sector W (with vertex at $0 \in \mathbb{C}$). In the latter case, the element in question is holomorphic on W and of the form $z \mapsto z^a(1 + u(z))$ where $a \in \mathbb{C}$ and u is defined on a neighborhood of $0 \in \mathbb{C}$ with $u(0) = 0$. In fact, the generators of Γ that are not defined on a neighborhood of $0 \in \mathbb{C}$ are obtained by means of singular loops passing through Siegel singularities of \mathcal{F} in \mathcal{K} what leads to the general form mentioned above. Among these elements we have $f(z) = z^\lambda$ with $\lambda > 1$.

In view of the preceding we can suppose without loss of generality that either h is defined on a neighborhood of $0 \in \mathbb{C}$ or is of the form $h(z) = z^a(1 + u(z))$.

Claim 1. h does not preserve the orbits of $\phi_{i\mathcal{X}}$.

Proof of Claim 1. Since h does not preserve \mathcal{X} up to a constant factor, it follows that

$$h^*\mathcal{X} = c_h(z)\mathcal{X}$$

where $c_h(z)$ is a non-constant holomorphic function on its domain. By construction the purely imaginary flow associated to $h^*\mathcal{X}$ is contained in $\bar{\Gamma}$ since so is $\phi_{i\mathcal{X}}$. Denoting by \mathcal{X}_1 the (real) vector field associated to this (real) 1-parameter group, we see that \mathcal{X}_1 is \mathbb{R} -linearly independent with \mathcal{X} at points in the (non-empty) open set where c_h takes values in $\mathbb{C} \setminus \mathbb{R}$. This proves the claim. \square

Next we have:

Claim 2. The vector fields \mathcal{X} and \mathcal{X}_1 do not commute. Besides $[\mathcal{X}, \mathcal{X}_1]$ is not constant.

Proof of Claim 2. Recall that $\mathcal{X} = z\partial/\partial z$ whereas $\mathcal{X}_1 = c_h(z)z\partial/\partial z$. Hence

$$[\mathcal{X}, \mathcal{X}_1] = -z^2 c'_h(z).$$

Since we have assumed that c_h is not constant we conclude that $c'_h(z)$ is not identically zero and the claim follows. \square

Consider a (“generic”) point p at which \mathcal{X} , \mathcal{X}_1 are \mathbb{R} -linearly independent and where $[\mathcal{X}, \mathcal{X}_1](p) = (a + ib)\partial/\partial z$ with $a \neq 0$. The existence of these points follows from Claim 2. For fixed small reals s, t , consider the local diffeomorphism D^{st} fixing p that is obtained as follows: points close to p are moved by following the (purely imaginary) flow of \mathcal{X} during a time s and then we compose this with the purely imaginary flow of \mathcal{X}_1 during a time t . Finally we still compose the mentioned map with the (always purely imaginary) flow of \mathcal{X} (and then \mathcal{X}_1) during a time s' close to s (resp. t' close to t) so as to have p as a fixed point of the resulting map D^{st} . Clearly the map D^{st} belongs to $\bar{\Gamma}$ for small s, t . It must therefore preserve the transverse measure over Σ identified to a neighborhood of $0 \in \mathbb{C}$. Therefore to derive a contradiction proving the statement, it suffices to show that p is a hyperbolic fixed point provided that s, t are small enough. To check

this simply note that the derivative of D^{st} at p is given by

$$1 - st[\mathcal{X}, \mathcal{X}_1](p) + o(s^2 + t^2) = 1 - st(a + ib) + o(s^2 + t^2).$$

Since $\|1 - st(a + ib)\| = 1 - 2sta + s^2t^2(a^2 + b^2)$ with $a \neq 0$, it follows that the norm of the derivative of D^{st} at p is different from 1 provided that s, t are very small. This finishes the proof of the proposition. \square

EXAMPLES

Let us close this paper with two classes of non-trivial examples of foliations for which Theorems A and B in the Introduction can immediately be applied. In the sequel consider a foliation \mathcal{F} on a surface M along with a global meromorphic form ω defining \mathcal{F} . Unless otherwise stated we choose ω so that its divisor of zeros and poles $(\omega)_\infty \cup (\omega)_0$ is disjoint from the finite set formed by the singularities of \mathcal{F} . To further simplify the discussion, suppose also that all singularities of \mathcal{F} have at least one eigenvalue different from zero. In view of Seidenberg's theorem our construction can easily be adapted to include foliations with degenerate singularities so that we shall not worry about them here.

Obviously the simplest way to ensure the existence of \mathcal{H} -trajectories with infinite length consists of eliminating the singular points of \mathcal{H} that give rise to either future endpoints or to past endpoints for trajectories of \mathcal{H} . Under the above assumptions we have:

- All points giving rise to past-ends for trajectories of \mathcal{H} are contained in $(\omega)_\infty$.
- All points giving rise to future-ends for trajectories of \mathcal{H} are contained in the union of $(\omega)_0$ with the singular set $\text{Sing}(\mathcal{F})$ of \mathcal{F} .

In addition for a singular point of \mathcal{F} to yield future ends for \mathcal{H} -trajectories the ratio between its eigenvalues must belong to \mathbb{R}_+^* .

To apply our theorems we need to have $(\omega)_\infty \neq \emptyset$ for otherwise ω is holomorphic and hence closed. Thus this particular choice of ω does not yield an associated foliation \mathcal{H} . Hence a natural idea is to eliminate the possibility future-ends for the trajectories of \mathcal{H} . With this idea in mind we shall provide our first class of examples.

EXAMPLE 1. Foliation with $(\omega)_0 = \emptyset$ and singularities whose eigenvalues have quotients in $\mathbb{C} \setminus \mathbb{R}_+^*$.

We note that the class of foliations above include those with singularities in the Siegel domain. Saddle-node singularities can also be authorized for \mathcal{F} . Suppose that T is a diffuse closed current invariant by \mathcal{F} . The first important remark to be made about T concerns its co-homology class in M . In fact, under the conditions regarding the singularities of \mathcal{F} the following holds:

$$[T].[T] = 0$$

ie. the self-intersection of T vanishes. The proof of the above equation is easy and essentially amounts to the fact that Siegel singularities cannot contribute non-trivially for the self-intersection of a diffuse current, see for example [Br]. Naturally, as seen in Section 4, all singularities of \mathcal{F} lying in the support of T must belong to the Siegel domain since the support of a diffuse current as above cannot contain either hyperbolic singularities or saddle-nodes, cf. Section 4. This case is therefore of little interest for surfaces, such as $\mathbb{C}P(2)$, whose Picard group is cyclic. However for surfaces with larger Picard group the condition about the self-intersection of T conveys less information.

For example let M be an affine elliptic $K3$ surface in $\mathbb{C}^3 \subset \mathbb{C}P(3)$ (i.e. the closure of M , still denoted by M , in $\mathbb{C}P(3)$ is a $K3$ -surface). Indeed, we can choose M to be the usual Fermat's quartic. Next consider a polynomial vector field X tangent to M and having isolated singularities. The vector field X induces a foliation \mathcal{F} over $M \subset \mathbb{C}P(3)$ whose singularities satisfy the above conditions modulo choosing X "generic". By exploiting the triviality of the canonical bundle of M , we can easily find a 1-form ω on M defining \mathcal{F} and having empty divisor of zeros. Furthermore the divisor of poles of ω coincides with the hyperplane section of M and, in general, ω is not closed. Now if T is a closed current invariant by \mathcal{F} , the condition that $[T].[T] = 0$ says that $[T]$ is the cohomology class of an elliptic fiber of M . However there is *a priori* no reason to conclude the existence of any compact leaf for \mathcal{F} unless Theorem A is used.

EXAMPLE 2. Foliations on $\mathbb{C}P(2)$ with singularities whose quotient of eigenvalues belong to \mathbb{R}_+ .

A disadvantage of the construction employed in Example 1 is that, after all, it depends on the fact that only Siegel singularities can appear on the support of a diffuse (closed positive) invariant current. Similarly it has been known since long that a foliation of $\mathbb{C}P(2)$ all of whose singularities are hyperbolic cannot admit a diffuse current as above. To be able to make a significant progress with respect to these well-known results, it is interesting to allow the foliation to have simple singularities of type "irrational focus". In fact, these singularities may contribute non-trivially to self-intersection of T and they do not contradict the diffuse nature of T . The example below is intended to show that our theorem can be used to provide new results in this context.

To simplify suppose that this happens at a single singularity p with eigenvalues $1, 1$ (ie. in local coordinates about p \mathcal{F} is given by the radial vector field $x\partial/\partial x + y\partial/\partial y$). The remaining singularities of \mathcal{F} being hyperbolic, of Siegel type or saddle-nodes. The existence of p prevents us from concluding that $[T].[T] = 0$. Nonetheless we claim the following:

Claim. If \mathcal{F} admits a closed current T , then it must also possess an algebraic curve provided that the degree of \mathcal{F} is at least 3.

Proof. Choose affine coordinates so that p belongs to the corresponding line at infinity Δ . In the affine \mathbb{C}^2 we choose a polynomial form ω representing \mathcal{F} and such that its components have only trivial common factors. Viewed as a meromorphic 1-form on $\mathbb{C}P(2)$, the divisor of zeros of ω is empty whereas ω has poles of order $d - 1$ over Δ where d stands for the degree of \mathcal{F} . Thanks to Theorem A to prove the claim it suffices to show that all the trajectories of the resulting foliation \mathcal{H} have infinite length. In turn, for this, it is enough to show that p does not provide future endpoints for these trajectories. To check the latter claim, consider local coordinates (x, y) about p , $\{x = 0\} \subset \Delta$, where $\omega = v(x, y)x^{1-d}(xdy - ydx)$ with $v(0, 0) \neq 0$. If we blow-up \mathcal{F} at p , the new foliation $\tilde{\mathcal{F}}$ has no longer singularities over the exceptional divisor which, in fact, is transverse to $\tilde{\mathcal{F}}$. In standard (x, t) coordinates for the blow-up the pull-back of ω is given by

$$v(0, 0)x^{3-d}dt.$$

In particular the pull-back of ω does not have zeros over the exceptional divisor since $v(0, 0) \neq 0$ and $d \geq 3$. Therefore the initial singularity p does not provide future endpoints for the trajectories of \mathcal{H} . The claim is proved. \square

Naturally the eigenvalues of \mathcal{F} at p may be supposed to have only the form $1, \lambda$, where $\lambda \in \mathbb{R}_+^*$. Besides, if the degree of \mathcal{F} is at least 4, we can allow the existence of two (rather than one) irrational focus singularities for \mathcal{F} . Several other combinations of these ideas can be used to provide new results on the structure of invariant curves for foliations as above.

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